KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXIV

Nos. 4 and 5.

President: Prof. F. A. F. C. WENT.

Secretary: Prof. L. BOLK.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXX).

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(Communicated at the meeting of October 29, 1921).

§ 1. *Introduction.*

In the theory of curvature of a V_m, imbedded in a V_n, a great number of theorems can be deduced with the aid of the quantity of order three $\overset{3}{\mathbf{H}}$ without use of the RIEMANN-CHRISTOFFEL quantity of order four of the V_m [1]) and the V_n, $\overset{4}{\mathbf{K}}$ and $\overset{4}{\mathbf{K}}'$. We have developed these theorems in detail in another paper [2]) and we will indicate them here only as far as is absolutely necessary. Other theorems, e.g. those concerning the invariants of deformation, depend on the mutual relations of the quantities $\overset{4}{\mathbf{K}}$ and $\overset{4}{\mathbf{K}}'$. This paper will treat the most important theorems of this kind for the most general case, which is not yet investigated sufficiently.

§ 2. *V_m in V_n, absolute, relative and normal curvature of a congruence.*

Suppose a congruence $\mathbf{i}$ [3]) in a V_m in V_n. The fundamental tensor of the V_n be ${}^1\mathbf{g} = \mathbf{aa} = \mathbf{bb} = \ldots$, the fundamental tensor of the V_m being ${}^1\mathbf{g}' = \mathbf{a}'\mathbf{a}' = \mathbf{b}'\mathbf{b}' = \ldots$. In the V_n choose n mutual orthogonal congruences $\mathbf{i}_1, \ldots, \mathbf{i}_n$, in such a way that V_m is built up by curves $\mathbf{i}_1, \ldots, \mathbf{i}_m$. We suppose that the suffixes i, j, k, l may get the values $1, \ldots, n$; a, b, c, d the values $1, \ldots, m$ and e, f, g, h the values $m+1, \ldots, n$.

When

$$\mathbf{a}'' = \sum_e a_e \mathbf{i}_e \qquad (1)$$

hence

---

[1]) V_p means in this paper a p-dimensional manifold, whose linear element is represented by the square root of a general quadratic differential form, S_p means such a manifold with constant RIEMANN-curvature, R_p such a manifold with euclidean linear element.

[2]) 22 . 1.

[3]) The notations used in this paper are developed in detail in 21. 1, shorter also in 21.2.

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}''. \tag{2}$$

we have

$$^2\mathbf{g} = \sum_j a_j \mathbf{i}_j = \sum_a a_a \mathbf{i}_a + \sum_e a_e \mathbf{i}_e = {}^2\mathbf{g}' + \mathbf{a}'' \mathbf{a}''. \tag{3}$$

It is obvious that

$$a_a a_b = \begin{cases} 1 \text{ if } a = b \\ 0 \text{ ,, } a \neq b \end{cases}, \quad a_e a_f = \begin{cases} 1 \text{ if } e = f \\ 0 \text{ ,, } e \neq f \end{cases}, \quad a_a a_e = 0 \tag{4}$$

and thus

$$\mathbf{a}' \mathbf{a}'' = \mathbf{a}'' \mathbf{a}' = 0. \tag{5}$$

The quantity $\overset{2p}{\mathbf{g}'}$ is defined by

$$\overset{2p}{\mathbf{g}'} = \mathbf{a}_1' \ldots \mathbf{a}_p' \mathbf{a}_p' \ldots \mathbf{a}_1', \tag{6}$$

so that by complete transvection of $\overset{p}{\mathbf{v}}$ with $\overset{2p}{\mathbf{g}'}$ we get the V_m-component of $\overset{p}{\mathbf{v}}$. If then ∇ be the differential operator in the V_n, ∇' in the V_m, the equation holds:

$$\nabla' p = {}^2\mathbf{g}' \overset{1}{\cdot} \nabla p, \tag{7}$$

where p is an arbitrary scalar field in the V_m, and

$$\left.\begin{aligned} \nabla' \mathbf{v} &= \{\nabla' (\mathbf{a}' \overset{1}{\cdot} \mathbf{v})\} \mathbf{a}' = \{\nabla' (\mathbf{a} \overset{1}{\cdot} \mathbf{v})\} \mathbf{a}' = \\ &= {}^2\mathbf{g}' \overset{1}{\cdot} \{\nabla (\mathbf{a} \overset{1}{\cdot} \mathbf{v})\} \mathbf{a} \overset{1}{\cdot} {}^2\mathbf{g}' = \overset{4}{\mathbf{g}'} \overset{2}{\cdot} \nabla \mathbf{v}, \end{aligned}\right\} \tag{8}$$

where $\mathbf{v}$ is an arbitrary vector field in the V_m.

Hence:

If $\mathbf{v}$ *is a vectorfield in* V_m, $\nabla' \mathbf{v}$ *is the* V_m*-component* of $\nabla \mathbf{v}$.

The same holds, as is easy to see, for each field $\overset{p}{\mathbf{v}}$ in V_m.

The curvature vector $\mathbf{u} = \mathbf{i} \overset{1}{\cdot} \nabla \mathbf{i}$ of the congruence $\mathbf{i}$ in V_m with respect to V_n, or the *absolute curvature vector* of $\mathbf{i}$ can now be decomposed in the following way:

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{i} \overset{1}{\cdot} \nabla \mathbf{i} = \mathbf{i} \overset{1}{\cdot} \{\nabla (\mathbf{a} \ \mathbf{i})\} \mathbf{a} = \mathbf{i} \overset{1}{\cdot} \{\nabla (\mathbf{a}' . \mathbf{i})\} \mathbf{a} = \\ &= \mathbf{i} \overset{1}{\cdot} \{\nabla (\mathbf{a}' . \mathbf{i})\} \mathbf{a}' + \mathbf{i} \overset{1}{\cdot} \{\nabla' (\mathbf{a}' . \mathbf{i})\} \mathbf{a}'' = \\ &= \mathbf{i} \overset{1}{\cdot} \nabla' \mathbf{i} + \mathbf{i} \overset{1}{\cdot} (\nabla' \mathbf{a}') \overset{1}{\cdot} \mathbf{i} \mathbf{a}'' + \mathbf{i} \overset{1}{\cdot} (\nabla \mathbf{i}) \overset{1}{\cdot} \mathbf{a}' \mathbf{a}'' = \\ &= \mathbf{u}' + \mathbf{i} \mathbf{i} \overset{2}{\cdot} (\nabla \mathbf{a}') \mathbf{a}'' \end{aligned}\right\} \tag{9}$$

in which formula $\mathbf{u}'$ is the *relative curvature vector* of $\mathbf{i}$ with respect to V_m.

When we write:

$$\overset{3}{\mathbf{H}} = (\nabla' \mathbf{a}') \mathbf{a}'' \tag{10}$$

we have:

$$\mathfrak{u} = \mathfrak{u}' + \mathfrak{i}\,\mathfrak{i} \overset{2}{.} \overset{3}{\mathfrak{H}} = \mathfrak{u}' + \mathfrak{u}''. \tag{11}$$

In this equation $\mathfrak{u}'' = \mathfrak{i}\,\mathfrak{i} \overset{2}{.} \overset{3}{\mathfrak{H}}$ is a vector perpendicular to V_m, $\mathfrak{a}''$ containing but $\mathfrak{i}_{m+1}, \ldots, \mathfrak{i}_n$. We call $\mathfrak{u}''$ the *forced curvature vector* (erzwungener Krümmungsvektor) with respect to the V_n and $\overset{3}{\mathfrak{H}}$ the *curvature affinor* of the V_m with respect to the V_n [1]). We thus have the theorem:

The absolute curvature vector of a curve in the V_m is the sum of the relative curvature vector [2]) *and the forced curvature vector with respect to the V_m.*

$\mathfrak{u}'$ vanishes for a curve, geodesic in V_m, thus $\mathfrak{u}''$ is the absolute curvature vector of a curve, geodesic in V_m, with the same tangent vector as the given curve. If $\mathfrak{u}''$ vanishes, the curve is called *asymptotic line of first order* of the V_m. Hence a geodesic line in V_m is then and only then geodesic in V_n, when it is an asymptotic line of first order of the V_m.

§ 3. *Theorems concerning the quantity* $\overset{3}{\mathfrak{H}}$.

Because of

$$(\nabla\,\mathfrak{a})\,\mathfrak{a} = 0 \text{ [3])} \tag{12}$$

we also have

$$(\nabla\,\mathfrak{a})\,\mathfrak{a}'' = 0 \tag{13}$$

and thus we can give to $\overset{3}{\mathfrak{H}}$ the following form:

$$\overset{3}{\mathfrak{H}} = \overset{4}{\mathfrak{g}'} \overset{2}{.} (\nabla\,\mathfrak{a}')\,\mathfrak{a}'' = -\overset{4}{\mathfrak{g}'} \overset{2}{.} (\nabla\,\mathfrak{a}'')\,\mathfrak{a}'' = -\overset{4}{\mathfrak{g}'} \overset{2}{.} \nabla\,\mathfrak{a}''\,\mathfrak{a}''. \tag{14}$$

Since however

$$\nabla\,\mathfrak{a}''\,\mathfrak{a}'' = \nabla\,\overset{2}{\mathfrak{g}} - \nabla\,\overset{2}{\mathfrak{g}'} = -\nabla\,\overset{2}{\mathfrak{g}'}, \tag{15}$$

we can get another form for $\overset{3}{\mathfrak{H}}$:

$$\overset{3}{\mathfrak{H}} = \overset{4}{\mathfrak{g}'} \overset{2}{.} \nabla\,\overset{2}{\mathfrak{g}'}. \tag{16}$$

From (14) still another form can be deduced:

$$\overset{3}{\mathfrak{H}} = -\overset{4}{\mathfrak{g}'} \overset{2}{.} \nabla \sum_e \mathfrak{i}_e\,\mathfrak{i}_e = -\sum_e \overset{4}{\mathfrak{g}'} \overset{2}{.} \nabla\,\mathfrak{i}_e\,\mathfrak{i}_e = -\sum_e \overset{4}{\mathfrak{g}'} \overset{2}{.} (\nabla\,\mathfrak{i}_e)\,\mathfrak{i}_e. \tag{17}$$

---

[1]) $\overset{3}{\mathfrak{H}}$ corresponds with the expression Ω_{rtj} in Voss, 80, 1, with $b_{\alpha\beta rs}$ in Ricci 03, 2 and with $k^{(r)}_{fg}$ in Kühne, 04, 1.

[2]) Ricci, 02,1 calls this vector "curvatura normale relativa a V_n".

[3]) 21. 1 p. 71.

Thus the equation of an asymptotic line of first order

$$\mathbf{i}\,\mathbf{i}\,{}^2_{\cdot}\,\overset{3}{\mathbf{H}} = 0 \tag{18}$$

is equivalent to the $n-m$ equations:

$$\mathbf{i}\,\mathbf{i}\,{}^2_{\cdot}\,\nabla\,\mathbf{i}_e = 0. \tag{19}$$

Since $\mathbf{i}_e$ is perpendicular to V_m, $\nabla\mathbf{i}_e$ is symmetrical in $\mathbf{i}_a\mathbf{i}_b$, and hence we conclude from (17) that $\overset{3}{\mathbf{H}}$ is symmetrical in its first two ideal factors.

When each geodesic line in V_m is also a geodesic line in V_n, V_m is called *geodesic* in V_n. If $\overset{3}{\mathbf{H}} = 0$, this is certainly fulfilled. This condition however is also necessary, because $\mathbf{i}\,\mathbf{i}\,{}^2_{\cdot}\,\overset{3}{\mathbf{H}}$ vanishes in this case for every choice of $\mathbf{i}$, and $\overset{3}{\mathbf{H}}$ is symmetrical in the first two ideal factors.

We thus have the theorem:

A V_m in the V_n is then and only then geodesic, when $\overset{3}{\mathbf{H}}$ vanishes in every point of the V_m.

If $\overset{3}{\mathbf{H}}$ vanishes only in one point P of the V_m, the V_m is called *geodesic in this point.* This case occurs e.g. if the V_m is built up by geodesic lines of the V_n all going through P.[1])

If in a point of the V_m the curvature vector $\mathbf{i}\,\mathbf{i}\,{}^2_{\cdot}\,\overset{3}{\mathbf{H}}$ has the same direction for every choice of $\mathbf{i}$, the point is called *axial.* For $m = n-1$ all points are axial.

A special case occurs, when $\overset{3}{\mathbf{H}}$ has the form:

$$\overset{3}{\mathbf{H}} = {}^2\mathbf{g}'\,\mathbf{U}. \tag{20}$$

A point, where this occurs, is called an *umbilical point* of the V_m. $\mathbf{U}$ is called the *umbilical vector.* All curves through an umbilical point have in this point the same forced curvature vector, and this vector is equal to $\mathbf{U}$.

For $m = n-1$ $\overset{3}{\mathbf{H}}$ has the form:

$$\overset{3}{\mathbf{H}} = -\overset{4}{\mathbf{g}'}\,{}^2_{\cdot}\,(\nabla\,\mathbf{i}_n)\,\mathbf{i}_n = -\,{}^2\mathbf{h}\,\mathbf{i}_n. \tag{21}$$

The symmetrical quantity ${}^2\mathbf{h}$:

$${}^2\mathbf{h} = \overset{4}{\mathbf{g}'}\,{}^2_{\cdot}\,\nabla\,\mathbf{i}_n \tag{22}$$

---

[1]) Bianchi 99, 1, p. 572 calls a V_2 in the V_n in this case already a geodesic V_2.

is the *second fundamental tensor* of the V_{n-1} [1]). It is obvious that in general

$$\nabla \mathbf{i}_n = {}^2\mathbf{h} + \mathbf{i}_n \mathbf{u}_n , \tag{23}$$

where $\mathbf{u}_n = \mathbf{i}_n \,\lrcorner\, \nabla \mathbf{i}_n$ is the curvature vector of $\mathbf{i}_n$. If we choose $\mathbf{i}_n$ geodesic, the formula is simpler:

$${}^2\mathbf{h} = \nabla \mathbf{i}_n \tag{24}$$

$$\overset{3}{\mathbf{H}} = - {}^2\mathbf{h}\, \mathbf{i}_n = - (\nabla \mathbf{i}_n)\, \mathbf{i}_n . \tag{25}$$

The forced curvature vector

$$\mathbf{u}'' = - \mathbf{i}\, \mathbf{i} \,\lrcorner\, {}^2\mathbf{h}\, \mathbf{i}_n \tag{26}$$

has for $m = n-1$ always the direction of $\mathbf{i}_n$ and gets an extreme value in the principal directions of ${}^2\mathbf{h}$. Hence these principal directions are also the directions of principal curvature [2]). When $\mathbf{i}_a$, $a = 1, \ldots, n-1$ are unit vectors in these directions and R_a the principal radii of curvature, with positive sign when the curvature vector has the sense of $\mathbf{i}_n$, we have

$${}^2\mathbf{h} = - \sum_a \frac{1}{R_a} \mathbf{i}_a \mathbf{i}_a \tag{27}$$

and from this we get the theorem:

The degree of nullity [3]) *of the second fundamental tensor is equal to the number of vanishing principal radii of curvature.*

If all principal radii of curvature in a point of the V_{n-1} are equal, and only in this case, ${}^2\mathbf{h}$ is a multiple of the fundamental tensor of the V_m:

$${}^2\mathbf{h} = - \frac{1}{R} {}^2\mathbf{g}' \tag{28}$$

and $\overset{3}{\mathbf{H}}$ thus assumes the form (20). Hence the point is an umbilical point.

§ 4. *Relations between the* Riemann-Christoffel *affinors of the V_m and the V_n.*

For the V_n the Riemann-Christoffel affinor has the form [4]):

$$\overset{4}{\mathbf{K}} = 2 \{ (\nabla \frown \nabla)\, \mathbf{a} \}\, \mathbf{a} = 2 \{ \nabla (\mathbf{a} \,.\, \mathbf{c}) \frown \nabla (\mathbf{b} \,.\, \mathbf{c}) \} (\mathbf{a} \frown \mathbf{b}) \tag{29}$$

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[1]) We can find indeed: $h_{\lambda\mu} = {}^1/_2\, \mathbf{i}_n \,\lrcorner\, \nabla g_{\lambda\mu}$. Compare e.g. Bianchi 99. 1, p. 601 and Schouten—Struik 19. 1, p. 207; 19. 2, p. 601.

[2]) First defined by Kronecker 69, 1, V_{n-1} in R_n.

[3]) The degree of nullity of a tensor of second order in V_p is the degree of nullity of the matrix of the p^2 components.

[4]) Comp. 21.1, p. 73.

and for the V_m:

$$\overset{4}{K}' = 2\{\nabla'(a'.c) \frown \nabla(b'.c')\}(a' \frown b'). \tag{30}$$

Dividing **a**, **b** and **c** as in (2), we get:

$$\left.\begin{aligned} \overset{8}{g}' \overset{4}{:} \overset{4}{K} &= 2\overset{4}{g}' \overset{2}{:} \{\nabla(a'.c') \frown \nabla(b'.c')\}(a' \frown b') + \\ &+ 2\overset{4}{g}' \overset{2}{:} \{\nabla(a'.c') \frown \nabla(b''.c'')\}(a' \frown b') + \\ &+ 2\overset{4}{g}' \overset{2}{:} \{\nabla(a''.c'') \frown \nabla(b'.c')\}(a' \frown b') + \\ &+ 2\overset{4}{g}' \overset{2}{:} \{\nabla(a''.c'') \frown \nabla(b''.c'')\}(a' \frown b'). \end{aligned}\right\} \tag{31}$$

The first of the four terms on the right in (31) is equal to $\overset{4}{K}'$, the second and third terms are equal, because they pass into each other by changing **a** and **b**.

The three last terms can all be expressed in $\overset{3}{H}$. For on one side we have with respect to (14):

$$\overset{3}{H} = \overset{4}{g}' \overset{2}{:} (\nabla c')\, c'' = \overset{4}{g}' \overset{2}{:} \{\nabla(a'.c')\}\, a\, c'' = {}^{2}\overset{4}{g}' \overset{1}{:} \{\nabla(a'.c')\}\, a'\, c'' \tag{32}$$

and on the other side with respect to (5):

$$\left.\begin{aligned} \overset{3}{H} &= -\overset{4}{g}' \overset{2}{:} \{(\nabla a') \overset{1}{:} c\, a\, c\} = -\overset{4}{g}' \overset{2}{:} \{\nabla(a''.c)\}\, a'\, c = \\ &= -{}^{2}\overset{4}{g}' \overset{1}{:} \{\nabla(a''.c'')\}\, a'\, c = -{}^{2}\overset{4}{g}' \overset{1}{:} \{\nabla(b''.d'')\}\, b'\, d. \end{aligned}\right\} \tag{33}$$

When now we write:

$$\overset{3}{H} = H_1\, H_1\, H_2 = {}'H_1\, {}'H_1\, {}'H_2\,, \tag{34}$$

which is permitted, $\overset{3}{H}$ being symmetrical in the first two factors, we have

$$\left.\begin{aligned} (H_1 \frown {}'H_1)(H_1 \frown {}'H_1)(H_2 . {}'H_2) &= -\overset{4}{g}' \overset{2}{:} \{\nabla(a'.c') \frown \nabla(b''.d'')\}(a' \frown b')(c''.d'') \\ &= -\overset{4}{g}' \overset{2}{:} \{\nabla(a'.c') \frown \nabla(b''.c'')\}(a' \frown b') \end{aligned}\right\} \tag{35}$$

and also

$$\left.\begin{aligned} (H_1 \frown {}'H_1)(H_1 \frown {}'H_1)(H_2 . {}'H_2) &= \overset{4}{g}' \overset{2}{:} \{\nabla(a''.c'') \frown \nabla(b''.d'')\}(a' \frown b')(c'.d') \\ &= \overset{4}{g}' \overset{2}{:} \{\nabla(a''.c'') \frown \nabla(b''.c'')\}(a' \frown b'), \end{aligned}\right\} \tag{36}$$

hence we get:

$$\boxed{\overset{8}{g}' \overset{4}{:} \overset{4}{K} = \overset{4}{K}' - 2(H_1 \frown {}'H_1)(H_1 \frown {}'H_1)\, H_2 . {}'H_2}\,. \tag{37}$$

This is the *Gaussian curvature theorem*, generalized for V_m in V_n [1]).

When we write

$$\overset{4}{\mathbf{g}}' \overset{2}{.} \nabla \mathbf{i}_e = {}^2\mathbf{h}_e = \mathbf{h}_e \mathbf{h}_e = {}'\mathbf{h}_e {}'\mathbf{h}_e, \tag{38}$$

we can give to (37) on account of (17) the following form:

$$\overset{8}{\mathbf{g}}' \overset{4}{.} \overset{4}{\mathbf{K}} = \mathbf{K}' - 2 \sum_e (\mathbf{h}_e \frown {}'\mathbf{h}_e)(\mathbf{h}_e \frown {}'\mathbf{h}_e) \tag{39}$$

The second term on the right in (37) and (39) vanishes e.g., when the V_m is geodesic in P. Hence the theorem holds:

The RIEMANN-CHRISTOFFEL *affinor of a V_m in V_n, geodesic in a point, is in this point the V_m-component of the* RIEMANN-CHRISTOFFEL *affinor of the V_n.*

The second term on the right vanishes also in a V_m with only axial points, when the degree of nullity of the tensor $^2\mathbf{h}$, belonging to the favoured normal direction, is one.

If the V_n is an S_n (comp. p. 146):

$$\overset{4}{\mathbf{K}} = 2 K_0 {}^2\mathbf{g} \barwedge {}^2\mathbf{g}\ ^2) \tag{40}$$

and the V_m is geodesic in P, the term on the left in (37) and (39) passes into

$$\overset{8}{\mathbf{g}}' \overset{4}{.} \overset{4}{\mathbf{K}} = 2 K_0 {}^2\mathbf{g}' \barwedge {}^2\mathbf{g}'. \tag{41}$$

Hence $2(\mathbf{H} \frown {}'\mathbf{H}_1)(\mathbf{H}_1 \frown {}'\mathbf{H}_1)\mathbf{H}_2 . {}'\mathbf{H}_2$ in this case depends only on the linear element of V_m, the m-direction of the V_m in the considered point and on K_0. If the V_n is an S_n and if the V_m contains only umbilical points, we derive from (39) that the V_m is an S_m. This holds consequently also in the particular case that the V_m is geodesic.

§ 5. *Absolute, relative and forced curvature of a V_m in V_n.*

Transvecting (39) totally with $-\frac{1}{m(m-1)} \mathbf{a}'\mathbf{b}'\mathbf{b}'\mathbf{a}'$, we get:

$$-\frac{1}{m(m-1)} \overset{4}{\mathbf{g}}' \overset{4}{.} \overset{4}{\mathbf{K}} = K_0' + \frac{2}{m(m-1)} \overset{4}{\mathbf{g}}' \overset{4}{.} \sum_e (\mathbf{h}_e \frown {}'\mathbf{h}_e)(\mathbf{h}_e \frown {}'\mathbf{h}_e). \tag{42}$$

We call $K_0' = -\frac{1}{m(m-1)} \mathbf{a}'\mathbf{b}'\mathbf{b}'\mathbf{a}' \overset{4}{.} \overset{4}{\mathbf{K}}'$ *the absolute curvature* of the V_m, it is the curvature of V_m considered as a manifold for its

[1]) LIPSCHITZ, 70, 1 p. 292, V_m in V_n; Voss, 80, 1 p. 139, V_m in V_n; RICCI, 02, 1, p. 359, V_m in V_n; KÜHNE, 03, 1, p. 309, V_m in V_n; comp. also e.g. BIANCHI 99, 1, p. 602, V_{n-1} in V_n; SERVANT, 02, 3, p. 94, V_2 in R_4.

[2]) 21. 1 p. 76.

own. It is an invariant with eventually possible deformations of the V_m in the V_n.

The quantity on the left is the curvature of the V_m tangent to the V_m in P and geodesic in this point, in particular thus of the V_m built up by the geodesic lines of the V_n tangent to the V_m in P. We call this quantity *the forced curvature* K_z of the V_m:

$$K_z = -\frac{1}{m(m-1)} \overset{4}{\mathbf{g}'} \overset{4}{:} \overset{4}{\mathbf{K}}. \tag{43}$$

The last term with negative sign is the curvature of the V_m, if the linear element of the V_n were euclidean, or the *relative curvature* K_r of the V_m:

$$K_r = -\frac{2}{m(m-1)} \overset{4}{\mathbf{g}'} \overset{4}{:} \sum_e (\mathbf{h}_e \frown {}'\mathbf{h}_e)(\mathbf{h}_e \frown {}'\mathbf{h}_e). \tag{44}$$

Hence we have the theorem:

The absolute curvature of a V_m in V_n is an invariant of deformation and equal to the sum of the relative and the forced curvature.

When the V_n is an S_n, we have

$$K_z = -\frac{1}{m(m-1)} \overset{4}{\mathbf{g}'} \overset{4}{:} \overset{4}{\mathbf{K}} = K_s, \tag{45}$$

consequently K_z is independent of the situation and the linear element of the V_m. Hence in this case K_r is also an invariant of deformation. In the general case K_r is an invariant of deformation in P for all deformations of the V_m, with which $\overset{8}{\mathbf{g}'} \overset{4}{:} \overset{4}{\mathbf{K}}$ is an invariant, thus in particular for the deformations with which the m-direction of the V_m in P remains unaltered.

The relative curvature of the V_m with respect to the V_{m+1} built up by a direction $\mathbf{i}_e$ continued in some way in the V_n, is:

$$K_{re} = -\frac{2}{m(m-1)} \overset{4}{\mathbf{g}'} \overset{4}{:} (\mathbf{h}_e \frown {}'\mathbf{h}_e)(\mathbf{h}_e \frown {}'\mathbf{h}_e). \tag{46}$$

We thus have:

$$K_r = \sum_e K_{re}, \tag{47}$$

in words:

When we pass through a V_m in an arbitrary way $n-m$ mutual $\frac{1}{m+1}$-perpendicular V_{m+1}, the relative curvature of the V_m with

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[1]) Voss, 80, 1. p. 172.

[2]) The names absolute and relative curvature are introduced by Ricci, 02, 1, p. 361.

respect to the V_n is the sum of the relative curvatures with respect to these V_{m+1} [1]).

§ 5. *The relations of the relative curvature to the principal radii of curvature and the simplest invariants of deformation.*

For $m = n-1$ we have:

$$\left.\begin{aligned} K_r &= -\frac{2}{m(m-1)}\,\mathfrak{g}'\,{}^{4}_{\cdot}\,(\mathfrak{h}\frown{}'\mathfrak{h})\,(\mathfrak{h}\frown{}'\mathfrak{h}) \\ &= -\frac{2}{m(m-1)}\sum_{ab}\mathfrak{i}_a\mathfrak{i}_b\mathfrak{i}_b\mathfrak{i}_a\,{}^{4}_{\cdot}\,(\mathfrak{h}\frown{}'\mathfrak{h})\,(\mathfrak{h}\frown{}'\mathfrak{h}) \\ &= -\frac{2}{m(m-1)}\sum_{ab}\mathfrak{i}_a\mathfrak{i}_b\,(\mathfrak{i}_b\frown\mathfrak{i}_a)\,{}^{4}_{\cdot}\,\mathfrak{h}'\mathfrak{h}\,\mathfrak{h}'\mathfrak{h} \\ &= -\frac{1}{m(m-1)}\sum_{ab}(\mathfrak{i}_a\mathfrak{i}_b\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_n)(\mathfrak{i}_b\mathfrak{i}_a\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_n) - (\mathfrak{i}_a\mathfrak{i}_a\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_n)(\mathfrak{i}_b\mathfrak{i}_b\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_n) \end{aligned}\right\} \quad (48)$$

Choosing the $\mathfrak{i}_a$ in the principal directions of curvature, we have:

$$\left.\begin{aligned} \mathfrak{i}_a\mathfrak{i}_b\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_n &= 0 \quad , \quad a \neq b \\ \mathfrak{i}_a\mathfrak{i}_a\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_n &= -\frac{1}{R_a} \end{aligned}\right\} \quad (49)$$

and we get:

$$K_r = \frac{1}{m(m-1)}\sum_{a,b}^{a\neq b}\frac{1}{R_a}\frac{1}{R_b}. \quad (50)$$

in words:

The relative curvature of a V_{n-1} in a V_n is the mean value of the twofactorical products with different suffixes of the principal curvatures.

Hence for a V_{n-1} in S_n this sum is an invariant of deformation, for a V_{n-1} in V_n with $K_a = 0$ it is equal to the negative forced curvature.

For arbitrary values of m we have:

$$K_r = -\frac{1}{m(m-1)}\sum_{e}\sum_{a,b}^{a\neq b}(\mathfrak{i}_a\mathfrak{i}_b\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_e)(\mathfrak{i}_b\mathfrak{i}_a\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_e) - (\mathfrak{i}_a\mathfrak{i}_a\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_e)(\mathfrak{i}_b\mathfrak{i}_b\,{}^{2}_{\cdot}\,\nabla\mathfrak{i}_e). \quad (51)$$

Now choose the $\mathfrak{i}_a$ in different ways for each value of $\mathfrak{i}_e$ and each time in the directions of principal curvature with respect to the normal vector $\mathfrak{i}_e$.

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[1]) Killing, 85, 1, p. 247, V_m in S_n; Berzolari 98, 1, p. 697, V_m in S_n, (both authors use projections); Ricci, 02, 1, p. 361, V_m in V_n.

We then have

$$K_r = \frac{1}{m(m-1)} \sum_{e} \sum_{a,b}^{a \neq b} \frac{1}{R_{ea}} \frac{1}{R_{eb}}, \tag{52}$$

in words:

The relative curvature of a V_m in the V_n is the sum of the mean values of the twofactorical products of the principal curvatures with different suffixes with respect to $n-m$ arbitrary mutual perpendicular normal directions of the V_m.

Thus for a V_m in S_n this sum is an invariant of deformation and for a V_m in V_n it is invariant with all deformations with which $\overset{8}{\mathbf{g}}' \overset{4}{.} \overset{4}{\mathbf{K}}$ is invariant in P, in particular with all deformations with which the m-direction of the V_m in P remains unaltered. For a V_m in the V_n with $K_a = 0$ it is equal to the negative forced curvature.

§ 6. *Other invariants of deformation of a V_m.*

Now we consider the sum of the four-factorical products with different suffixes of the principal curvatures with respect to the normal $\mathbf{i}_e$. Since

$$\left.\begin{aligned}
\overset{8}{\mathbf{g}}' \overset{8}{.} \mathbf{h}_e^4 \frown \mathbf{h}_e^4 \frown &= \overset{8}{\mathbf{g}}' \overset{8}{.} (\mathbf{h}_e \frown {}'\mathbf{h}_e \frown {}''\mathbf{h}_e \frown {}'''\mathbf{h}_e)(\mathbf{h}_e \frown {}'\mathbf{h}_e \frown {}''\mathbf{h}_e \frown {}'''\mathbf{h}_e) = \\
&= \sum_{abcd} \mathbf{i}_a \mathbf{i}_b \mathbf{i}_c \mathbf{i}_d (\mathbf{i}_d \mathbf{i}_c \mathbf{i}_b \mathbf{i}_a) \overset{8}{.} \mathbf{h}_e {}'\mathbf{h}_e {}''\mathbf{h}_e {}'''\mathbf{h}_e \mathbf{h}_e {}'\mathbf{h}_e {}''\mathbf{h}_e {}'''\mathbf{h}_e = \\
&= \tfrac{1}{24} \sum_{abcd}^{abcd \neq} (\mathbf{i}_a \mathbf{i}_a \overset{2}{.} \nabla \mathbf{i}_e)(\mathbf{i}_b \mathbf{i}_b \overset{2}{.} \nabla \mathbf{i}_e)(\mathbf{i}_c \mathbf{i}_c \overset{2}{.} \nabla \mathbf{i}_e)(\mathbf{i}_d \mathbf{i}_d \overset{2}{.} \nabla \mathbf{i}_e) = \\
&= \tfrac{1}{24} \sum_{abcd}^{abcd \neq} \frac{1}{R_{ea} R_{eb} R_{ec} R_{ed}},
\end{aligned}\right\} \tag{53}$$

this sum is equal to $\overset{8}{\mathbf{g}}' \overset{8}{.} \mathbf{h}_e^4 \frown \mathbf{h}_e^4 \frown$. In the same way we can prove that for the sum $\sigma_{\alpha e}$ of the α-factorical products with different suffixes of the principal curvatures, $\alpha \leqq m$, with respect to the normal direction $\mathbf{i}_e$ the equation holds:

$$\sigma_{\alpha e} = (-1)^{\frac{\alpha(\alpha-1)}{2}} \overset{2\alpha}{\mathbf{g}}' \overset{2\alpha}{.} \mathbf{h}_e^{\alpha} \frown \mathbf{h}_e^{\alpha} \frown, \tag{54}$$

We now form the series of quantities:

$$\left.\begin{aligned}
\overset{6}{\mathbf{H}} &= (\mathbf{H}_1 \frown {}'\mathbf{H}_1)(\mathbf{H}_1 \frown {}'\mathbf{H}_1) \mathbf{H}_2 {}'\mathbf{H}_2 \\
\overset{9}{\mathbf{H}} &= (\mathbf{H}_1 \frown {}'\mathbf{H}_1 \frown {}''\mathbf{H}_1)(\mathbf{H}_1 \frown {}'\mathbf{H}_1 \frown {}''\mathbf{H}_1) \mathbf{H}_2 {}'\mathbf{H}_2 {}''\mathbf{H}_2 \\
&\vdots \\
\overset{3\alpha}{\mathbf{H}} &= \mathbf{H}_1^{\alpha} \quad \mathbf{H}_1^{\alpha} \frown \mathbf{H}_2^{\alpha}
\end{aligned}\right\} \tag{55}$$

When we want to write out the powers, we have to introduce equivalent factor systems $\mathbf{H}_1, \mathbf{H}_1, \mathbf{H}_2, {}'\mathbf{H}_1, {}'\mathbf{H}_1, {}'\mathbf{H}_2$ etc. $\overset{3\alpha}{\mathbf{H}}$ is symmetrical in the last α factors, just because all α systems are equivalent, and contains in these factors only the units $\mathbf{i}_{m+1}, \ldots, \mathbf{i}_n$. We then have:

$$\mathbf{h}_e^{\alpha\frown}\,\mathbf{h}_e^{\alpha\frown} = \overset{3\alpha}{\mathbf{H}} \overset{\alpha}{:} \mathbf{i}_e^{\alpha} \tag{56}$$

and:

$$\sigma_{\alpha e} = (-1)^{\frac{\alpha(\alpha-1)}{2}}\, \overset{2\alpha}{\mathbf{g}'}\, \overset{2\alpha}{:}\, \overset{3\alpha}{\mathbf{H}} \overset{\alpha}{:} \mathbf{i}_e^{\alpha}\,. \tag{57}$$

The mean value $\bar{\sigma}_{\alpha e}$ of the values of $\sigma_{\alpha c}$ belonging to all possible directions $\mathbf{i}_e$ can be obtained by transvecting $\overset{2\alpha}{\mathbf{g}'}\, \overset{2\alpha}{:}\, \overset{3\alpha}{\mathbf{H}}$ with the mean value of all quantities $\mathbf{i}_e^{\alpha}$. Now it can be easily calculated that this latter mean value vanishes for α odd and >1, and is equal to $({}^1\mathbf{g}-{}^1\mathbf{g}')^{\mu\frown}$ save a numerical factor for α even, $\alpha = 2\mu$. So we have e.g.:

$$M.value\ \mathbf{i}_e\,\mathbf{i}_e\,\mathbf{i}_e\,\mathbf{i}_e = \frac{3}{(m-n)^2 + 2\,(m-n)}\,({}^1\mathbf{g}-{}^1\mathbf{g}')^{2\frown} \tag{58}$$

$$M.value\ \mathbf{i}_e^6 = \frac{15}{(n-m)^3 + 6\,(m-n)^2 + 8\,(m-n)}\,({}^1\mathbf{g}-{}^1\mathbf{g}')^{3\frown}. \tag{59}$$

Since $\overset{3\alpha}{\mathbf{H}}$ is symmetrical in the last α factors, we thus have for $\alpha = 2\mu$:

$$\bar{\sigma}_{\alpha e} = \lambda_\mu\, \overset{4\mu}{\mathbf{g}'}\, \overset{4\mu}{:}\, \overset{6\mu}{\mathbf{H}}\, \overset{2\mu}{:}\, {}^1\mathbf{g}^\mu \tag{60}$$

where μ is a positive integer, λ_μ a numerical factor only depending on $n-m$, and in which expression ${}^1\mathbf{g}^\mu$ may be replaced by every permutation of the 2μ ideal factors of the μ factors ${}^1\mathbf{g}$.

Now $\overset{6\mu}{\mathbf{H}}\, \overset{2\mu}{:}\, {}^1\mathbf{g}^\mu$ can be built up by the ideal factors of $\overset{6}{\mathbf{H}}\, \overset{2}{:}\, {}^1\mathbf{g}$, but since on account of (37):

$$2\,\overset{6}{\mathbf{H}}\, \overset{2}{:}\, {}^1\mathbf{g} = -\,\overset{8}{\mathbf{g}'}\, \overset{4}{:}\, \overset{4}{\mathbf{K}} + \overset{4}{\mathbf{K}'}, \tag{61}$$

we see that also $\overset{6\mu}{\mathbf{H}}\, \overset{2\mu}{:}\, {}^1\mathbf{g}^\mu$ only depends on $\overset{8}{\mathbf{g}'}\, \overset{4}{:}\, \overset{4}{\mathbf{K}}$ and $\overset{4}{\mathbf{K}'}$. Hence $\bar{\sigma}_{\alpha e}$ only depends on ${}^1\mathbf{g}'$, $\overset{8}{\mathbf{g}'}\, \overset{4}{:}\, \overset{4}{\mathbf{K}}$ and $\overset{4}{\mathbf{K}'}$. Consequently it has the same invariance as K_r.

Now choose $\alpha + \beta$ even, $\alpha + \beta = 2\mu$ and consider the quantity

$$\overset{6\mu}{\mathbf{J}} = \mathbf{H}_1^{\alpha\frown}\,\mathbf{L}_1^{\beta\frown}\,\mathbf{H}_1^{\alpha\frown}\,\mathbf{L}_1^{\beta\frown}\,\mathbf{H}_2^{\alpha}\,\mathbf{L}_2^{\beta} \tag{62}$$

built up of $\overset{3\alpha}{\mathbf{H}}$ and $\overset{3\beta}{\mathbf{H}}$, $\mathbf{L}$ being equivalent with $\mathbf{H}$. When we want to write out the powers we have to introduce α equivalent systems $\mathbf{H}_1, \mathbf{H}_1, \mathbf{H}_2, {}'\mathbf{H}_1, {}'\mathbf{H}_1, {}'\mathbf{H}_2$ etc. and in the same way β systems $\mathbf{L}_1, \mathbf{L}_1, \mathbf{L}_2, {}'\mathbf{L}_1, {}'\mathbf{L}_1, {}'\mathbf{L}_2$ etc. We then have

$$\mathfrak{h}_e^{\alpha\frown}\,\mathfrak{h}_e^{\beta}\,\mathfrak{h}_e^{\alpha\frown}\,\mathfrak{h}_e^{\beta\frown} = {}^{6\mu}\mathfrak{J}^{2\mu}\,\mathfrak{i}_e^{2\mu} \tag{63}$$

and

$$\sigma_{\alpha e}\,\sigma_{\beta e} = (-1)^{\frac{\alpha(\alpha-1)}{2}+\frac{\beta(\beta-1)}{2}}\,{}^{4\mu}\mathfrak{g}'\,{}^{4\mu}\mathfrak{J}^{6\mu}\,\mathfrak{J}^{2\mu}\,\mathfrak{i}_e^{2\mu}. \tag{64}$$

Hence the mean value of all quantities $\sigma_{\alpha e}\,\sigma_{\beta e}$, $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ is equal to

$$\overline{\sigma_{\alpha e}\,\sigma_{\beta e}} = (-1)^{\frac{\alpha(\alpha-1)}{2}+\frac{\beta(\beta-1)}{2}}\,{}^{4\mu}\mathfrak{g}'\,{}^{4\mu}\mathfrak{J}^{6\mu}\,\mathfrak{J}^{2\mu}\,(\mathfrak{i}_e^{2\mu})_m, \tag{65}$$

in which $(\mathfrak{i}_e^{2\mu})_m$ is the mean value of all quantities $\mathfrak{i}_e^{2\mu}$.

In order to transform this expression, we consider the ideal quantity

$$\mathfrak{P} = ({}^{(\beta)}\mathfrak{a}\,\ldots\,{}^{\prime}\mathfrak{a})\,({}^{(\alpha)}\mathfrak{b}\,\ldots\,{}^{\prime}\mathfrak{b}), \tag{66}$$

in which the $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent ideal factors of ${}^{2}\mathfrak{g}$:

$${}^{2}\mathfrak{g}' = {}^{\prime}\mathfrak{a}\,{}^{\prime}\mathfrak{a} = \ldots = {}^{(\beta)}\mathfrak{a}\,{}^{(\beta)}\mathfrak{a} = {}^{\prime}\mathfrak{b}\,{}^{\prime}\mathfrak{b} = \ldots = {}^{(\alpha)}\mathfrak{b}\,{}^{(\alpha)}\mathfrak{b}. \tag{67}$$

This quantity arises from

$${}^{(\beta)}\mathfrak{a}\,\ldots\,{}^{\prime}\mathfrak{a}\;\;{}^{(\alpha)}\mathfrak{b}\,\ldots\,{}^{\prime}\mathfrak{b} \tag{68}$$

by replacing two definite systems of α, resp. β ideal factors by their alternating product. Such an operation is called a *simple alternation* [1]. In the general case a simple alternation ${}_{s_1\ldots s_t}A$ replaces t definite systems of $s_1, s_2, \ldots, s_t$ factors by their alternating product. $s_1, \ldots, s$ is called the *permutation-number* of the alternation. Every alternation is apparently a multiple sum of permutations. The alternations with different permutation-number can be ordered according to their permutation-number and then get a suffix on the right. So e.g. we have for $2\mu = 6$:

$$\left.\begin{array}{l} {}_0A = A_1\;,\; {}_2A = A_2\;,\; {}_{2,2}A = A_3\;,\; {}_{3,2}A = A_4\;,\; {}_3A = A_5\;,\; {}_{3,2}A = A_6 \\ {}_{2,3}A = A_7\;,\; {}_4A = A_8\;,\; {}_{4,2}A = A_9\;,\; {}_5A = A_{10}\;,\; {}_6A = A_{11} \end{array}\right\} \tag{69}$$

The number of the different possible permutation-numbers be k. For $2\mu = 6$ we have therefore $k = 11$. The different alternations with the same permutation-number can be distinguished by a suffix high on the right. So there are e.g. for $2\mu = 6$ 15 different alternations A_6:

$$A_6^{\,1}\,\ldots\,A_6^{\,15}.$$

The sum of all alternations A_u with the same permutation-number divided by its number, is called the *general alternation* $\overline{A}_u$. Now we can show that there are k operators ${}_uI$ such that

$${}_uI\,{}_vI = \begin{cases} {}_uI & \text{if } u = v \\ 0 & ,,\; u \neq v \end{cases} \tag{70}$$

and

[1] For the development of the theory of these operators and their application to the expansion of quantities and forms in series see 19, 3, English 19, 4.

$$ {}_uI\,A_v = A_v\,{}_uI = \begin{cases} 0 \text{ if } v > u \\ \neq 0 \;,, \; v \leqq u \end{cases} \tag{71} $$

the sum of all ${}_ul$ being the identical operator. $\bar{A}_v$ can be written as a multiple sum of the operators ${}_vI, \ldots, {}_kI$:

$$ \bar{A}_v = \sum_w^{v,\ldots,k} \frac{1}{\delta_{vw}}\,{}_wI \tag{72} $$

When now we write

$$ {}_{\beta,\alpha}A = A_v \quad , \quad {}_{\beta,\alpha}\bar{A} = \bar{A}_v \tag{73} $$

and

$$ {}_{\mu.2}\bar{A} = \bar{A}_u, \tag{74} $$

we have for $\alpha + \beta > 4$ and for $\alpha + \beta = 4$, $\alpha \neq \beta$ certainly $u < v$, while $u = v$ for $\alpha + \beta = 4$, $\alpha = \beta$. For $u < v$ however we have:

$$ \overset{2\mu}{\mathbf{P}} = A_v \overset{2\mu}{\mathbf{P}} = A_v \sum_w^{v,\ldots,k} {}_wI \overset{2\mu}{\mathbf{P}} = \sum_w^{v,\ldots,k} {}_wI\,A_v \overset{2\mu}{\mathbf{P}} = \bar{A}_u \sum_w^{v,\ldots,k} \delta_{uw}\,{}_wI\,A_v \overset{2\mu}{\mathbf{P}} . \tag{75} $$

while for $\alpha + \beta = 4$ and $\alpha = \beta$ we have $u = v$:

$$ {}_{\beta,\alpha}A = {}_{2.2}A. \tag{76} $$

Hence $\overset{2\mu}{\mathbf{P}}$ is for $\alpha + \beta \geqq 4$ always a sum of quantities all alternating in μ systems of two factors. Their number is the number of simple alternations ${}_{\mu.2}A$. We consider of these quantities an arbitrary one and call the corresponding operator, of which this quantity is built up from $\overset{2\mu}{\mathbf{P}}$: O_γ. Then:

$$ \left.\begin{aligned} &O_\gamma = A_\mu^\nu \sum_w^{v,\ldots,k} \delta_{uw}\;{}_wI\,A_v \text{ if } \alpha + \beta > 4 \text{ or } \alpha + \beta = 4,\ u \neq v \\ &O_\gamma = A_\mu^\nu \text{ if } \alpha + \beta = 4,\ u = v. \end{aligned}\right\} \tag{77} $$

We then have:

$$ \left.\begin{aligned} &({}^{(\beta)}\mathbf{a} \ldots \overset{\frown}{\ldots} {}'\mathbf{a})\,({}^{(\alpha)}\mathbf{b} \ldots \overset{\frown}{\ldots} {}'\mathbf{b})\,({}^{(\beta)}\mathbf{a} \ldots \overset{\frown}{\ldots} {}'\mathbf{a})\,({}^{(\alpha)}\mathbf{b} \ldots \overset{\frown}{\ldots} {}'\mathbf{b}) = \sum_\gamma (O_\gamma \overset{2\mu}{\mathbf{P}})\,\overset{2\mu}{\mathbf{P}} \end{aligned}\right\} \tag{78} $$

Since, however, the $\mathbf{a}$ and $\mathbf{b}$ are all equivalent, we apparently have for each simple alternation

$$ (A_v \overset{2\mu}{\mathbf{P}})\,\overset{2\mu}{\mathbf{P}} = \overset{2\mu}{\mathbf{P}}\,(A_v \overset{2\mu}{\mathbf{P}}) = (A_v \overset{2\mu}{\mathbf{P}})\,(A_v \overset{2\mu}{\mathbf{P}}) \tag{79} $$

and in consequence, since ${}_wI$ can be written as a multiple sum of alternations:

$$ (O_\gamma \overset{2\mu}{\mathbf{P}})\,\overset{2\mu}{\mathbf{P}} = (O_\gamma \overset{2\mu}{\mathbf{P}})\,(O_\gamma \overset{2\mu}{\mathbf{P}}). \tag{80} $$

Now we have according to (65) and (66)

$$\left.\begin{aligned}\overline{\sigma_{\alpha e}\,\sigma_{\beta e}} &= (-1)^{\frac{\alpha(\alpha-1)}{2}+\frac{\beta(\beta-1)}{2}}\overset{2\mu}{P}\,\overset{2\mu}{P}\,\overset{4\mu}{:}\,H_1^{\alpha}\,L_1^{\beta}\,H_1^{\alpha}\,L_1^{\beta}\,H_2^{\alpha}\,L_2^{\beta}\,\overset{2\mu}{:}\,(i_e^{2\mu})_m \\ &= \lambda_\mu\,\overset{2\mu}{P}\,\overset{2\mu}{P}\,\overset{4\mu}{:}\,H_1^{\alpha+\beta}\,H_1^{\alpha+\beta}\,H_2^{\alpha+\beta}\,\overset{2\mu}{:}\,{}^2g^{\mu} = \\ &= \sum_{\gamma}\lambda_\mu\,(O_\gamma\overset{2\mu}{P})\,O_\gamma\overset{2\mu}{P})\,\overset{4\mu}{:}\,H_1^{\alpha+\beta}\,H_1^{\alpha+\beta}\,H_2^{\alpha+\beta}\,\overset{2\mu}{:}\,{}^2g^{\mu}\end{aligned}\right\}\quad(81)$$

in which formula ${}^2g^{\mu}$ may be replaced in each term by every arbitrary permutation of ideal factors of the μ factors ${}^2g$, each term now being on its own symmetrical in the last 2μ factors. Since further

$$O_\gamma\overset{2\mu}{P}\,\overset{2\mu}{:}\,H_1^{\alpha+\beta} = O_\gamma\overset{2\mu}{P}\,\overset{2\mu}{:}\,A_\mu^{\nu}\;H_1^{\alpha+\beta}\qquad(82)$$

and A_u is an alternation with μ systems of two factors, we can build up every term of $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ by the ideal factors of μ quantities $\overset{6}{H}\,\overset{2}{:}\,{}^2g$ and 2μ quantities ${}^2g'$. Hence $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ depends only on ${}^2g'$, $\overset{8}{g'}\,\overset{4}{:}\,\overset{4}{K}$ and $\overset{4}{K'}$.

Hence we have got the following theorem.

In a definite point P of a V_m in V_n we consider the sums $\sigma_{\alpha e}$ of the α-factorical products of the principal curvatures with different suffixes with respect to a definite normal direction i_e of the V_m. We then form the mean values $\bar{\sigma}_{\alpha e}$ and $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$, $\alpha+\beta \geqq 4$ of all quantities $\sigma_{\alpha e}$ resp. $\sigma_{\alpha e}\sigma_{\beta e}$ with respect to all possible normal directions. Then $\bar{\sigma}_{\alpha e}$ resp. $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ vanish for α resp. $\alpha+\beta$ odd, and for α, resp. $\alpha+\beta$ even they depend only on the linear element of the V_m and on the V_m-component $\overset{8}{g'}\,\overset{4}{:}\,\overset{4}{K}$ of the Riemann-Christoffel *affinor $\overset{4}{K}$ of the V_n. Hence these quantities are invariants of deformation for a V_m in S_n and for a V_m in V_n they are invariant with all eventually possible deformations of the V_m in V_n that leave $\overset{8}{g'}\,\overset{4}{:}\,\overset{4}{K}$ unaltered, in particular with all deformations with which the m-direction of the V_m in P remains unaltered* [1]).

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[1]) Lipschitz has first proved (70, 1) the invariance of the $\bar{\sigma}_{\alpha e}$ for α even and for V_m in R_n, afterwards (76, 1) that of $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ for $\alpha+\beta$ even and $\geqq n+1$. Killing has (85, 1) given the proof for all $\bar{\sigma}_{\alpha e}$ and $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ for V_m in S_n, he also gives the geometrical interpretation of these expressions. Maschke has (06, 1) proved the invariance of σ_{n-1} for V_{n-1} in R_n with the aid of his symbolic calculus, Bates (11, 1) that of σ_α $\alpha > 1$ for V_{n-1} in R_n. Bates has not succeeded in expressing the $\bar{\sigma}_\alpha$ in ${}^2g'$, $\overset{4}{K}$ and $\overset{8}{g'}\,\overset{4}{:}\,\overset{4}{K}$ for V_m in V_n, though he finds expressions for [illegible]. These, however, contain still the $n-m$ functions that, equalled to zero, form the equations of the V_m.

For a V_{n-1} in the V_n $\overline{\sigma_{\alpha e}}$ passes for α even into σ_α and $\overline{\sigma_{\alpha e}\,\sigma_{\beta e}}$ passes for $\alpha+\beta$ even into $\sigma_\alpha\,\sigma_\beta$. Hence in this case all quantities σ_α for $n>3$ and all quantities σ_α, $\alpha>1$ for $n=3$ are only dependent on the linear element of the V_{n-1} and on $\overset{8}{\mathbf{g}}' \overset{4}{:} \overset{4}{\mathbf{K}}$. This theorem can be proved in a simpler way for $n>3$. After (39) $^2\mathbf{h}$ is the vector-tensor belonging to the bivector-tensor $\frac{1}{2}(\overset{4}{\mathbf{K}}' - \overset{8}{\mathbf{g}}' \overset{4}{:} \overset{4}{\mathbf{K}})$. Now a vector-tensor is then and only then uniquely determined by its corresponding bivector-tensor, when its degree of nullity is n, $n-1$ or $n-2$. For $n>3$ $^2\mathbf{h}$ is thus in general uniquely determined by $\overset{4}{\mathbf{K}}' - \overset{8}{\mathbf{g}}' \overset{4}{:} \overset{4}{\mathbf{K}}$ and consequently also all principal curvatures and all quantities σ_α. From this follows that a V_{n-1} in R_n or S_n, in which $\overset{4}{\mathbf{K}}' - \overset{8}{\mathbf{g}}' \overset{4}{:} \overset{4}{\mathbf{K}}$ only depends on the $(n-1)$-direction of the V_{n-1} in P, cannot in general be deformed [1]). We can observe that for the same reason a V_m in R_n or S_n with only axial points cannot in general be deformed in such way that its points remain axial. In a deformation of a V_m in V_n also an axial point remains in general not axial.

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Physics. — "*On the calculation of the molecular quadrupole-moments from the equation of state.*" By Prof. W. H. KEESOM. (Communication N°. 9 from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated at the meeting of September 24, 1921).

§ 1. *Introduction.* When the potential in the field of the electric charges in a rotational symmetric homopolar molecule is developed in a series of powers of r^{-1}, the first term may be regarded as being due to a zonal quadrupole.

As far as is evident from the investigation of the equation of state especially for hydrogen, the molecular attraction in diatomic homopolar gases may be regarded to a first approximation as due to such a quadrupolar term in the field of force of the molecule, at least for a high temperature and in a diluted gaseous state. That finally an attraction results must be ascribed to two causes: 1st that two molecules when approaching each other will try to direct each other in such a way that attracting forces arise between them, and 2nd that in two approaching molecules the charges are displaced by their mutual influence thus that this gives rise to an attraction.

As far as this DEBIJE[1]) and the author[2]) agree. Their opinions

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[1]) P. DEBIJE. Physik. ZS. **22**, p. 302, 1921. Very interesting is the application made by DEBIJE of these considerations to monatomic gases, where the mutually directing influence of the molecules mentioned under 1st becomes zero, so that only the attraction due to the polarisation of the molecules mentioned under 2nd remains. See for this also F. ZWICKY, Physik. ZS. **22**, p. 449, 1921. As to this, we may remark however the following. The application of the above to a quadrupole term in the field of the monatomic molecules gives us a mean value of the potential energy of two such molecules proportional with r^{-8} (a dipole term would give r^{-6}), while on the contrary the observations for argon are more in favour of r^{-4} or r^{-5} (hydrogen below the BOYLE-point r^{-4}), see Leiden Suppl. N°. 26 § 3, these Proceedings October 1912, p. 643.

[2]) Comm. N°. 6*b*, these Proc. Vol. XXIII, N°. 6, 1920. Physik. ZS. **22**, p. 129, 1921.

differ however on the magnitudes of the quadrupole moments [1]) that are to be ascribed to the molecules of the gases in question (hydrogen, oxygen and nitrogen) on account of the data of the equation of state.

The author has deduced these quadrupole moments [2]) from the experimentally found second virial coefficients for the above mentioned gases at selected temperatures by comparing them with the dependency of the second virial coefficient on the temperature as deduced for fixed quadrupoles and thus without attending to the displacement of the charges mentioned under 2nd. Afterwards [3]) he has proved that at those temperatures the influence of the displacibility of the charges is only small compared with that of the attraction due to the fixed quadrupoles. From this it is evident, that the values for the quadrupole moments derived from the experimental data do not undergo considerable changes by the displacibility of the charges.

Following another way of calculation however DEBIJE finds considerably higher values for the quadrupole moments and on several grounds he prefers these higher values.

In this communication will be shown that the author cannot adhere to this opinion and by further calculation he will investigate the influence of the polarisability of the molecules on the value found for the quadrupole moment. Then, as might be expected, this influence will be found to be small.

§ 2. *Rectification.* In the first place it may be mentioned here, that at the last moment a calculation error has been made in my former calculation of the quadrupole moment of hydrogen. From the values for σ and v given in Leiden Suppl. N°. 39*a* § 5 we find for the quadrupole moment of the hydrogen molecule:

$$\mu_2 = 1{,}17 \times 10^{-26} \quad [\text{e.s.e.} \times \text{cm.}^3],$$

instead of the value that was given there [4]).

As by his calculation of the quadrupole moment of hydrogen DEBIJE found the value

$$\mu_2 = 2{,}14 \times 10^{-26},$$

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[1]) For a molecule which is not rotationally symmetric (and which has no dipole-moment) this is replaced by a mean quadrupole moment, comp. DEBIJE l. c.

[2]) Leiden Suppl. N°. 39*a*. These Proc. Vol. XVIII, p. 636. These Comm. N°. 6*a*, these Proc. Vol. XXIII N°. 6 p. 939.

[3]) l.c. p. 162, note 2.

[4]) In note 1 p. 15 l.c. we must read therefore $0{,}70 \times 10^{-8}$ instead of $0{,}92 \times 10^{-8}$. As to the remark in § 1 of Comm. N°. 6*a*, these Proc Vol. XXIII p. 940 on the agreement between the values of the quadrupole moment deduced from the equation of state and from the model of BOHR-DEBIJE, this loses its sense.

the two methods of calculation are evidently leading to considerably different results for hydrogen too.

§ 3. His opinion that the polarisation of the molecules caused by their mutual influence, is of great importance, is based by DEBIJE principally on the values of the coefficients of the *mean* reduced equation of state. This mean equation of state has been given by KAMERLINGH ONNES for a systematic summary and discussion of the material of observations and of the deviations from the law of corresponding states. At the same time, however, he declared emphatically [1]) that for no substance this mean reduced equation of state will coincide with the real reduced equation of state. It would only do so, when the substances the data of which have been used in the derivation of the mean reduced equation of state, strictly obeyed the law of corresponding states. Now this is by no means the case, especially not for hydrogen compared with nitrogen and oxygen, which gases gave the data for the higher reduced temperatures, which are of most interest for our question. This is a. o. evident from the following fact. When for the great volumes and for the dominion of reduced temperatures corresponding to the temperature interval in which AMAGAT has made measurements on O_2 and N_2 we want to make the reduced equations of state of H_2 correspond with those of O_2 and N_2, we must choose as critical reduction temperature for H_2 [2]) 43, whereas however the critical temperature of H_2 is 33. Now in the mean reduced equation of state the reduced second virial coefficient $\mathfrak{B}$ is represented as a function of the reduced temperature t that is obtained by combining the values taken from H_2 with those of O_2 and N_2, etc. and reduced as well for H_2 as for O_2 and N_2 by means of the experimental values of T_k. And it is evident that this function can show a quite different character from that which corresponds to the behaviour of each of the substances.

For the discussion of questions as those considered here, we do better not to use the mean reduced equation of state.

It would be preferable to start from the *special* reduced equation of state of hydrogen [3]). In the above question however the special equation of state would lead to trustworthy results only then, when it was fitted to high temperatures. But this is not the case.

[1]) See e.g. Leiden Comm. N°. 74 § 4, 1901.

[2]) H. KAMERLINGH ONNES and C. BRAAK. Leiden Comm. N°. 97*b*, p. 39. H. KAMERLINGH ONNES and W. H. KEESOM. Die Zustandsgleichung. Math. Enz. V 10. Leiden Suppl. N°. 23, note 399.

[3]) See Leiden Comm. N°. 109*a*, § 7, 1909.

The special equation of state deduced from measurements of KAMERLINGH ONNES and BRAAK gives e.g.

$$B_\infty = 0{,}000893,$$

a value higher than might be expected from direct graphical extrapolation of the individual values of B from the measurements of KAMERLINGH ONNES and BRAAK. Therefore this special equation of state too would also give a too high value for the quadrupole moment when the method of calculation of DEBIJE was used.

It seems to me preferable to proceed as was done in my preceding communications viz. to work out the theoretically deduced development of B until [1]) it extends over a sufficiently wide dominion, to compare then this development with the experimental data and to deduce in this way e.g. the value of the quadrupole moment.

§ 4. *Further development of the second virial coefficient for spherical polarisable quadrupole molecules.* In order to take into consideration the influence of the polarisability of the molecules in the deduction of the quadrupole moment, the development started in Comm. N°. 6*b* § 3 has been extended by a few terms. Thereto the formulae given in Leiden Suppl. N°. 39*a* may be of use. By multiplying equation (11) from the paper by

$$\chi = \sin^4\theta_1 + \sin^4\theta_2 + 4\cos^4\theta_1 + 4\cos^4\theta_2 \quad . \quad . \quad . \quad . \quad (1)$$

(Comm. N°. 6*a* equation (10)), we find, following the notations of Leiden Suppl. N°. 39*a*:

$$[\psi^n\chi] = 2\pi[A^n\chi] + \binom{n}{2}[A^{n-2}B^2\chi][\cos^2\varphi] + \binom{n}{2}[A^{n-2}C^2\chi][\cos^2 2\varphi] +$$

$$+ \binom{n}{3}\binom{3}{1}[A^{n-3}B^2C\chi][\cos^2\varphi\cos 2\varphi] \text{ etc. } . \quad . \quad . \quad (2)$$

With

$$\sin^4\theta_1 + 4\cos^4\theta_1 = A_1^2 + 4B_1^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(see Leiden Suppl. N°. 39*a* equation (9)) we find:

$$[A^p B^{2q} C^r\chi] = 2^{p+8q+1}[A_1^p B_1^{2q} C_1^r]\{[A_1^{p+2} B_1^{2q} C_1^r] +$$

$$+ 4[A_1^p B_1^{2q+2} C_1^r]\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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[1]) Of course I acknowledge the objections made by DEBIJE in § 5 of his cited paper. They show that the results obtained in the above mentioned way can only be accepted with some reserve as has been specially stated in Comm. N°. 6*a* § 4. On account of all this accurate measurements on the second virial coefficient at higher temperatures are very necessary. In the meantime however it seems to me of some importance that such a good agreement with the experimental data that are at our disposition is obtained by using the mentioned simplifying suppositions (quadrupolar action, no quantum deviations as yet, collisions as of solid spheres).

Applying then equations (14), (15) and (16) of Leiden Suppl. N°. 39*a* and substituting in equation (16) of Comm. N°. 6*b*, we finally obtain:

$$B = \tfrac{1}{2} n \cdot \tfrac{4}{3} \pi\sigma^3 \Big\{ 1 - 1.0667\,(h\nu)^2 + 0.1741\,(h\nu)^3 - 0.4738(h\nu)^4 + 0.6252(h\nu)^5 \ldots$$
$$- 2{,}4\,\frac{a}{\sigma^3} h\nu\,[1 + 1\ 0667\,(h\nu)^2 - 0.3641\,(h\nu)^3 + 0.2267\,(h\nu)^4 \ldots] \Big\} \quad . \quad (5)$$

§ 5. *Hydrogen.* Substituting for hydrogen $\frac{a}{\sigma^3} = 0{,}0640$ (Comm. N°. 6*b* § 4) we find:

$$B = \tfrac{1}{2} n \cdot \tfrac{4}{3} \pi\sigma^3 \{ 1 - 0.1536\,h\nu - 1.0667\,(h\nu)^2 + 0.0103\,(h\nu)^3 -$$
$$- 0.4179\,(h\nu)^4 + 0.5904\,(h\nu)^5 - 0.2360\,(h\nu)^6 +$$
$$+ 0.1355\,(h\nu)^7 - 0.1079\,(h\nu)^8 + 0.0593\,(h\nu)^9 \ldots \} \quad . \quad . \quad . \quad . \quad (6)$$

This gives for the JOULE-KELVIN point of inversion for small densities (comp. Leiden Suppl. N°. 39*a* § 4):

$$h\nu_{inv\,(\rho=0)} = 0.503 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From this we derive for B expressed in terms of $\frac{T}{T_{inv(\rho=0)}} = t_{(inv)}$:

$$B = B_\infty \{ 1 - 0.0773\,t_{(inv)}^{-1} - 0.2699\,t_{(inv)}^{-2} + 0.01311\,t_{(inv)}^{-3} -$$
$$- 0.02675\,t_{(inv)}^{-4} - 0.01901\,t_{(inv)}^{-5} - 0.00382\,t_{(inv)}^{-6} +$$
$$+ 0.00110\,t_{(inv)}^{-7} - 0.00042\,t_{(inv)}^{-8} + 0.00011\,t_{(inv)}^{-9} \ldots \} \quad . \quad . \quad . \quad . \quad (8)$$

$\frac{T}{T_{inv\,(\rho=0)}}$	$\frac{B}{B_\infty}$	$\frac{B}{B_{inv}}$
0.6	0.160	0.246
0.75	0.426	0.650
1	0.655	1
1.5	0.829	1.266
2	0.895	1.366
3	0.944	1.411
4	0.963	1.470

In this way we reach an accuracy of about 1 % (of B_∞ at $T = 0{,}6\ T_{inv\,(\rho=0)}$). The preceding table gives some of the calculated values.

In fig. 1 these values have been plotted together with the

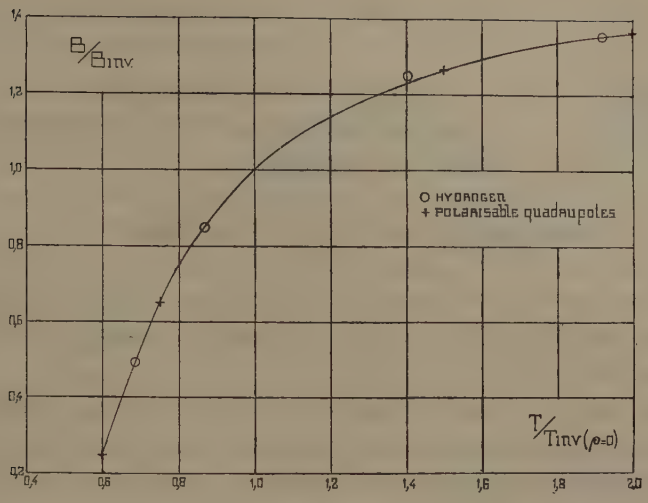

Fig. 1.

experimental data according to KAMERLINGH ONNES and BRAAK. Over the whole range of temperature that is considered, the agreement proves to be very good.

As in Leiden Suppl. N°. 39*a* § 5 we find further:

for σ the same value as was given there, but

$$v = 1.34 \times 10^{-14}$$

and the quadrupole moment

$$\mu_2 = 1.10 \times 10^{-26} \quad [\text{e.s.e.} \times \text{cm.}^2].$$

As to the quadrupole moment of H_2 we thus really find, as was expected in Comm. N°. 6*b* § 4, a value deviating only little from that deduced by the calculations of Leiden Suppl. N°. 39*a* (see this Comm. § 2).

Further we deduce still easily for μ_2 the following relation

$$\mu_2^2 = 9.19 \times 10^{-17} \cdot \sigma^5 \, T_{inv\,(\rho=0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For the reasons discussed in § 3 this equation seems to me preferable to equation (6) of DEBIJE l.c.

§ 6. *Oxygen and nitrogen.* For oxygen we may also apply equation (8). For nitrogen the coefficient will be a little different.

For the moment however we omit a more detailed calculation of the quadrupole moments of these gases as these may only be deduced with little accuracy because of the relatively small temperature interval in which for these gases data on B are at our disposal (Comm. N°. 6*a* § 2). For this reason the small alteration introduced by attending to the polarisability of the molecules would be of no importance.

Astronomy. — *"On a change with the declination in the personal equation in meridian-observations."* By C. H. HINS. (Communicated by Prof. W. DE SITTER).

(Communicated at the meeting of September 24, 1921).

In the preliminary reduction of the R.A. in the current programme of the Leyden meridian circle, the question arose as how to deal with the clock rate.

The first difficulty that arises is that the registering clock KNOBLICH is not the same as the principal clock of the observatory, HOHWÜ 17.

We shall therefore first give a brief account of how the reduction was made in former observations.

Before the beginning and after the end of the observations the beats of HOHWÜ 17 were registered for one and a half minute, with intervals of three seconds, on the slips upon which the KNOBLICH clock registers every two seconds. In this way the value of the difference of time Kn.—H. 17 was obtained at two epochs. Even when these comparisons were made three or more times in one evening, they were always represented by a straight line. (See PANNEKOEK, Annals of the Leyden Observatory Vol. X part I).

At regular intervals of about ten days determinations of time were made which gave the daily rate of the clock HOHWÜ 17. This "rough" rate observed in the interval between two determinations of time was reduced by the formula:

$p = \textit{daily rate} - 0^s,0153\,(b - 760^{m.m}) + 0^s,0263\,(t° - 10°) - 0\,,37\,(t - t')$

to the so called *reduced daily rate* p at $t = 10°$, $b = 760^{m.m}$ and $t - t' = 0$.

In the above formula

b represents the mean barometer in the interval.

t the mean temperature in the interval.

$t - t'$ the mean difference of the temperatures at the top and the bottom in the case of the clock.

daily rate: the rough rate as explained above.

From this *reduced daily rate* the actual rate of H. 17 during the observations was derived by the above formula and the observed values of b, t and $t - t'$.

All transits expressed in the time of the Kn. clock could be reduced to H. 17 time by means of the interpolated difference Kn.—H. 17, or, which comes to the same thing, from the observed relative rate

Kn.—H. 17 and the calculated absolute rate of H. 17 the absolute rate of the Kn. clock was derived.

By this method the observed transits of the fundamental stars give only the mean clock correction at a mean sidereal time and did not give any data for finding the rate of the clock.

The question now arose, whether it would not be possible, excluding the clock H. 17, to derive from the registered transits of the fund. stars not only the mean clock correction, but also the rate, both expressed in time of the Kn. clock.

In order to arrive at a solution of this question I here give the postulates underlying each of the methods.

I: (by means of H. 17).

1. The times of transits of the fund. stars must be sufficiently accurate to give a reliable clock correction by their mean.

2. The relative rate Kn.—H 17 obtained must be accurate, from which it follows that:

a. The rate of the Kn. clock must be constant during the observations.

b. The registering of the beats of H. 17 must be accurate.

c. There must not be a systematic difference between the registration of these beats before and after three or four hours of observations.

3. The rate of the H. 17 clock must conform to the formula given above, not only in the mean for several days, but must do so without any retardation, in other words it must react immediately to every change of temperature and barometer.

II: (without the H. 17 clock).

1. The transits of the fundamental stars must have sufficient accuracy to give by their mean a reliable clock correction and at the same time a reliable clock rate.

2. The rate of the Kn. clock must be constant during the observations.

A comparison of the two groups of postulates shows that II, 2 and I, 2*a* are the same.

The only difference between II, 1 and I, 1 is that in the second method the observations of the transits of fund. stars require a higher degree of refinement and the question is reduced therefore to the following:

What conditions must be laid down for observations to make them suitable for the deduction of a clock rate?

In the first place it is desirable that the stars of the observation programme should all be included between fundamental stars, so

that nowhere an extrapolation of the clock rate takes place; the night's work therefore should always begin and end with a number of fund. stars.

In the second place it is necessary that the fund. stars, or rather their groups, be at a sufficient distance from each other, so that the accidental errors in the times of transits shall have as little influence as possible.

Theoretically it is always possible to fulfil these requirements by a well chosen programme of observations. Only on nights interrupted by clouds it lies outside the power of the observer to fulfil them.

It is, therefore, necessary to examine more closely the accuracy of the times of transits obtained.

The clock corrections as derived from the separate stars are affected by two kinds of errors, accidental and systematic.

Of the influence of the first an idea can be obtained by computing the mean error of the time of transit derived from the transits of the separate threads (in Leyden numbering eleven). Generally speaking this error will not exceed $\pm 0^s.024$. If we had two groups of four fund. stars separated by an interval of two hours, the mean error of the clock rate, due exclusively to the accidental errors would thus be about

$$\pm \frac{\sqrt{0.012^2 + 0.012^2}}{120} = \pm 0^s.00014.$$

The second category of errors plays, however, at least as important a part and it is therefore necessary to make the times of transit as completely independent of them as possible.

Under these errors we include:

1. The magnitude equation.

By a suitable use of gratings it is possible to confine this error within narrow limits. What remains must be removed as much as possible by deriving the magnitude equation of the various observers and correcting for it.

2. The errors in the assumed positions of the fundamental stars. Since it is practically impossible to confine oneself to the small number of accurately established fundamental stars, it is necessary to use as fund. stars some that are of doubtful value, so that these errors must have a great influence not only on the mean clock correction, but even more on the rate of the clock as derived from the observations (see the large probable errors given by Boss' P. G. C. for various stars for the epoch 1910,0).

3. Personal errors of the different observers.

Assuming that the error mentioned under 1. be removed, I shall

endeavour to avoid as much as possible those mentioned under 2. and 3., beginning with 3. From the whole of the observations of all the observers, freed from their individual personal errors, it will then perhaps be possible to estimate those given under 2.

A brief account of the Leyden programme may precede. The stars of the programme are, as far as possible, divided into zones, whence in view of the somewhat restricted material (1600 stars distributed over 24 hours of right ascension and declination $-2°$. to $52°$, and the distribution in R.A. still very irregular) it was impossible to take the zones very narrow.

In general the zones are chosen with the limiting declinations of 0°—20°, 20°—30°, 30°—40° and 40°-zenith, but it was often necessary to include in a zone stars of slightly different declination. The fund. stars are so chosen that they lay below, in and above the zone so that their mean declination was as much as possible at the middle of the zone.

As, therefore, fundamental stars differing by 25° in declination are sometimes observed on the same night, the question arises, whether the different observers registered stars with such different declinations, i. e. of different velocities, all in the same way.

If a systematic difference of this kind existed, it would be desirable to reduce the times of transit of the separate stars to a hypothetical star with a declination equal to the mean declination of the fund. stars used. The times of transit of the programme stars could then be reduced also to the same hypothetical star, and by a purely differential reduction the results of the programme stars would be freed from this systematic error.

For this purpose the separate nights of the different observers are treated in the following way:

Let observer X on one night observe n fund. stars.

Times of transit $T_1 \ldots T_n$, mean T

Declinations $\delta_1 \ldots \delta_n$, mean δ

Clock correction derived from each star $a_1 \ldots a_n$, mean a.

As unknowns may be taken the rate of the clock $= x^s$ per minute and the possible influence of the declination on the time of transit $= + y^s$ per 1° deviation from the mean declination.

Every night gives n equations of the form:

$$(T - T_i)\, x^s + (\delta - \delta_i)\, y^s = (a^s - a_i^s) \qquad i = 1 \ldots n$$

From these n equations the unknowns x and y have been solved by least squares.

The x as found in this solution can be regarded as a first approximation to the rate of the clock.

We will now consider the values of y, which are obtained from different nights for the same observer.

Observer H.

The table below gives the results for y from 39 nights during the years 1920 and 1921. The list is arranged according to increasing mean declination of the fund. stars used.

1st column date of observation.
2nd „ mean declination of fund. stars used.
3rd „ result for y.

1920 March 21	3°.9	+0^s.00147	1920 Febr. 8	21°.8	+0^s.00065
„ „ 29	3°.9	+0.00123	„ „ 15	23°.9	+0.00274
„ May 23	10°.0	+0.00263	„ March 21	27°.6	+0.00151
„ „ 13	11°.8	−0.00160	„ Febr. 21	29°.3	+0.00304
1921 Febr. 23	13°.2	+0.00235	1921 March 3	29°.4	+0.00319
1920 „ 14	14°.4	−0.00163	1920 Febr. 11	29°.9	+0.00268
„ July 14	16°.3	+0.00250			
„ „ 15	16°.3	+0.00307			
„ May 18	18°.8	−0.00030			
1920 Jan. 21	30°.6	+0^s.00181	1920 May 16	40°.4	+0^s.00312
1921 Febr. 20	30°.7	+0.00283	„ „ 13	41°.0	+0.00387
1920 „ 17	32°.5	+0.00544	1919 Dec. 15	41°.6	+0.00863
1921 March 10	32°.8	+0.00169	1920 Jan. 25	41°.8	+0.00597
1920 Aug. 14	34°.4	+0.00162	„ March 21	43°.0	+0.00519
1921 Febr. 14	34°.8	+0.00317	„ Febr. 23	43°.2	+0.00620
1920 April 23	35°.9	+0.00430	„ Jan. 14	43°.5	+0.00642
„ Febr. 6	35°.9	+0.00343	„ Febr. 20	43°.6	+0.00638
„ May 3	35°.9	+0.00145			
„ „ 21	36°.5	+0.00149			
„ April 16	37°.9	+0.00247			
„ March 29	38°.4	+0.00356			
„ „ 18	38°.9	+0.00203			
„ April 10	39°.2	+0.00322			
„ Febr. 26	39°.7	+0.00465			
„ „ 19	39°.9	+0.00504			

It is clear by the predominating of the positive sign, that the systematic error looked for exists, but also, that the error is not constant in the different declinations, and shows a marked increase for increasing declination.

The following table gives the mean values of y, for the zones according to which the programme stars are divided:

Zône	0°—20°	12°,1	$+0^s,00108 \pm 61$
,,	20°—30°	27°,0	$0\ ,00230 \pm 42$
,,	30°—40°	35°,9	$0\ ,00301 \pm 33$
,,	40°—zenith	42°,3	$0\ ,00572 \pm 60$

As the velocity of the stars is proportional to sec. δ, I have endeavoured to represent the resulting values of y by a formula of the form:

$$y = A \frac{d}{d\delta} \sec \delta = A \tan \delta \sec \delta.$$

The four means given above, give four equations of condition for solving the quantity A. Each equation has been given a weight equal to the number of nights which have contributed to the equation. I find $A = + 0^s.00401$.

y is thus represented by the formula:

$$y = + 0^s.00401 \tan \delta \sec \delta.$$

The residuals between the observed and calculated values of y for the four zones are as follows:

$$-0^s,00020, \qquad -0^s,00001, \qquad +0^s,00057. \qquad -0^s,00079$$

The correction for the time of transit of a star with decl. δ_i, when the mean decl. of the fund. stars is δ, now becomes:

$$z = \frac{0^s.00401}{bg\ 1^\circ} \int_{\delta}^{\delta_i} \tan \delta \sec \delta\, d\delta = 0^s.230\ (\sec \delta_i - \sec \delta)$$

If $\delta_i > \delta$ the time of the transit must be corrected by a positive quantity.

If $\delta_i < \delta$ the time of the transit must be corrected by a negative quantity.

in other words:

Stars with high decl. are registered too early.

,, ,, small ,, ,, ,, late.

Observer Z.

List of results for y, obtained and arranged in the same way as for observer H.

1920 March 22	5°.0	$+0^s.00159$	1920 June 10	31°.2	+0 .00339
„ Febr. 23	11°.1	+0.00167	„ Febr. 3	32°.2	+0.00222
„ May 12	11°.3	−0.00013	„ June 8	32°.5	+0.00314
„ Febr. 18	14°.0	+0.00077	„ Aug. 27	32°.6	+0.00547
„ May 15	18°.4	+0.00095	„ Febr. 24	34°.3	−0.00017
„ „ 21	18°.4	+0.00041	„ „ 23	36°.8	+0.00535
1920 Febr. 12	22°.4	−0.00275	1920 June 21	40°.0	+0.00132
„ „ 5	23°.6	+0.00377	„ Aug. 17	42°.0	+0.00315
„ Oct. 28	25°.6	+0.00288	„ Jan. 9	42°.2	+0.01195
„ Nov. 19	28°.0	+0.00209	„ Febr. 19	43°.2	+0.01142
„ „ 8	28°.6	+0.00505			

The separate results give the following means:

Zone	0°—20°	13°,0	$+ 0^s,00088 \pm 28$
„	20°—30°	25°,6	$+ 0^s,00221 \pm 133$
„	30°—40°	33°,3	$+ 0^s,00324 \pm 86$
„	40°—zenith	41°,8	$+ 0^s,00696 \pm 276$

Owing to the smaller number of nights, the mean error is much greater, although the same influence is fairly marked.

Treated as above, y can be represented by the formula:

$$y = + 0^s.00497 \tan \delta \sec \delta.$$

The residuals $O-C$ for the four zones are:

$$-0^s,00030, \quad -0^s,00043, \quad -0^s,00068, \quad +0^s,00100$$

The correction z becomes:

$$z = + 0^s.285 (\sec \delta_i - \sec \delta).$$

For observer Z. we find the same phenomenon, that stars of high decl. are registered too early, those with small decl. too late.

Observer G.

List of the separate results, arranged as before.

1920 Febr. 17	4°.9	$+0^s.00022$
1921 Jan. 14	6°.7	−0.00158
1920 March 4	9°.6	+0.00260
„ „ 26	9°.7	−0.00018
„ Febr. 6	11°.3	0.00261
„ „ 15	11°.9	+0.00070
1921 „ 24	13°.3	+0.00090
„ „ 26	13°.3	+0.00143
1920 May 14	13°.4	−0.00075
1920 Febr. 18	22°.4	+0.00113
„ May 28	27°.2	−0.00174
„ June 9	30°.1	+0.00068
1921 March 15	30°.7	−0.00461
„ Febr. 22	31°.2	−0.00201
1920 „ 3	32°.0	−0.00269
„ May 19	33°.8	−0.00072
1921 March 11	34°.2	−0.00263
1920 Jan. 23	33°.9	+0.00297
„ July 17	34°.4	−0.00043
„ March 18	34°.9	+0.00036
1920 Febr. 7	35°.7	$+0^s.00123$
„ „ 26	35°.9	+0.00095
1921 „ 19	36°.2	−0.00207
1920 May 10	36°.7	+0.00105
„ „ 9	40°.3	+0.00285
„ June 15	41°.0	$+0^s.00012$

As the different declinations are spread more irregularly over the separate nights, I have arranged the y's in somewhat different groups.

Zone 0°—20° 10°,5 + $0^s.00008 \pm 53$
„ 20°—35° 31°.3 — $0^s.00088 \pm 64$
„ 35°— zenith 37°.6 + $0^s.00069 \pm 50$

Since these means are all nearly equal to or smaller than their m.e., we may conclude that for observer G. no influence of the declination on the time of transit exists.

The explanation of the effect may perhaps be sought in the fact, that the observers H. and Z. endeavour to make the contact of the signal key coincide with the moment that the star passes a thread, which is in agreement with their own statement as to their method of registering, while the observer G. begins with the movement of registering at the moment that he observes the star to be bisected by a thread.

That actually this difference in the method of registering exists between G. and Z. is confirmed by special observations, made for the determination of their relative personal equation, which is found to be positive in the sense G.—Z.

The observers H. and Z., therefore, will be influenced in their registering by the velocity $c.\sec.\delta$, with which the star *approaches* the thread, the observer G. will be independent of it.

As it is always possible in zone-observations with differential reduction to limit the influence of the magnitude-equation by using different gratings, while in practice it is often necessary to take the range of the zones relatively large, it will always be desirable for an observer who is aware that he follows the first method of registration, to test his observations of transits for a dependence on the declination. Perhaps we may see in this result once more an argument in favour of the impersonal micrometer.

Finally we give a table, which shows that the correction reaches quantities, which are by no means negligible.

Mean decl. fund. stars	Decl. programst.	Correction to the time of transit	
		Observer H.	Observer Z.
20°	10°	−0s.011	−0s.014
20°	15°	− 0.007	− 0.008
20°	25°	+ 0.009	+ 0.011
20°	30°	+ 0.021	+ 0.026
30°	20°	− 0.021	− 0.026
30°	25°	− 0.012	− 0.016
30°	35°	+ 0.015	+ 0.019
30°	40°	+ 0.035	+ 0.043
40°	30°	− 0.035	− 0.043
40°	35°	− 0.019	− 0.024
40°	45°	+ 0.025	+ 0.031
40°	50°	+ 0.058	+ 0.071

July 1921. *Leyden Observatory.*

Botany. — *"On Anti-phototropic Curvatures occurring in the coleoptiles of Avena"*. By Dr. C. E. B. BREMEKAMP. (Communicated by Prof. G. VAN ITERSON Jr.).

(Communicated at the meeting of October 29, 1921).

In a paper published some years ago [1]) I discussed at some length the relation between the nature of the phototropic curvatures occurring at the tip of the Avena coleoptiles and the way in which those organs are illuminated. I pointed out that the origin and the growth of those curvatures may be interpreted satisfactorily with the aid of a few suppositions which present themselves rather naturally. To this end it should be assumed that the organs in question consist of elements furnished with a photo-sensible system, and that the changes which the system undergoes under the influence of illumination are followed after some time by changes in the rate of growth. The value of the latter will surely not be equal to that of the changes which would occur if each element could react independently of the adjoining ones. As a matter of fact between the various elements a levelling will take place [2]). In the second place it should be assumed of those elements that they are equal, or approximately so, and lastly that the changes in the rate of growth are proportionate to the changes in the photo-sensibility.

The first-named supposition will no doubt remain hypothetical for the time being, since no method has been brought forward as yet o demonstrate the existence of those elements. However, the results obtained with illumination of isolated strips lend support to it. As to the second hypothesis it must be remarked that the elements alluded to are most likely not perfectly equivalent. The inner elements are circumstanced differently from those lying immediately

[1]) C. E. B. BREMEKAMP, Theorie des Phototropismus. Rec. d. Trav. bot. Néerlandais. Vol. XV. 1918.

[2]) It should be borne in mind, however, that owing to this levelling tensions must arise by which presumably the growth of the elements will be affected indirectly. In the case of the coleoptiles of Avena the value of these tensions will probably be insignificant, as the distribution of the light in the interior of the plant is, owing to the repeated reflections, rather diffuse here. It is a moot point, however, whether these tensions may be neglected also with organs such as the sporangium-bearer of Phycomyces, in which the differences of intensity in the interior of the plant are much larger.

under the surface [1]). It is highly questionable, therefore, whether their growth will be similarly influenced by the same causes. In case these differences are not too great, they may readily be neglected, since for making our calculations it is not necessary, as will be seen lower down, to know the reaction of each separate element, but only the joint-reaction of the elements in the anterior, and of those in the posterior side. If the light in these two halves were distributed quite symmetrically, these differences would not avail. In the given circumstances they also neutralize each other, however, for the greater part, and in this way much of their significance is lost. Whether the third hypothesis gives a correct representation of the relationship between the sensibility to light and the rate of growth, might be ascertained experimentally. To this end one would have to compare the sensibility to light and the rate of growth in the tip of omnilaterally lighted plants. Such experiments have, however, not been made thus far.

In order to calculate the curvatures on the basis of these hypotheses we have to know how the light is distributed over the two halves whose antagonism brings about the curvature, what changes the sensibility to light undergoes in those halves, and what is the relation between the changes in the rate of growth and those in the sensibility to light.

The distribution of the light over the two antagonistic halves may be deduced from the position of either the optimum or the maximum of the curve which represents the strength of the curvature as a function of the energy. With the optimum the difference in sensibility in the two halves must be greatest, [2]) with the maximum smallest. As the difference in sensibility increases so long as the inclination of the curve representing the sensibility of the frontal part is steeper than that of the curve representing the sensibility of the posterior part, it will be greatest when the difference between the inclinations of the two curves vanishes, and it will be smallest when the sensibility in both parts is lost. In order to determine the changes of the sensibility in the anterior, and in the posterior side separately we,

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[1]) It is supposed here that the sensibility to light is distributed rather diffusely over the whole organ. This is, however, not certain. If it could be demonstrated that the sensibility was restricted to a single layer, the elements could be regarded as perfectly equal.

[2]) This is stated roughly. A given difference in sensibility will, as is shown in my paper, bring about a larger curvature according as the average sensibility is smaller. In this case, however, the mean sensibilities differ so little that this factor does not carry weight.

therefore, have to know 1st either the optimum or the maximum of the curve representing the strength of the curvature, and 2nd in what way the sensibility can be represented as a function of the energy.

When we have found out what changes the sensibility in the anterior, and in the posterior side undergoes, it will at once be clear how the light is distributed over those two halves. Since more data are available relative to the optimum than to the maximum I started in my calculation from those relative to the optimum. The accurateness of the latter is, however, also very small, so that the value found for the proportion should be taken only as a first approximation.

The changes in the sensibility to light could be deduced from the data obtained by Arisz[1]) in his experiments on phototropic disposition. As most important among those changes I may mention 1st that the sensibility undergoes a gradual decrease in consequence of the illumination and is ultimately entirely destroyed; the required quantity of light should, however, be supplied within a fixed time (cf. 2nd); 2nd that the sensibility is ere long augmented again, the rate of the increase being greater according as the intensity of the illumination is smaller, the sensibility reaching at last a higher value in weak illuminations than in strong ones; 3rd that this is also the case after the light has been put out and that in this case the sensibility regains at last its original value; and 4th that the velocity of the increase reaches its highest value only after some time has elapsed. In order to determine the relation between the changes of the sensibility to light and those of the rate of growth we need only to compare, in a given case, the changes undergone by the difference in sensibility in the antagonistic sides with the growth of the curvature.

From the foregoing it may be concluded: 1st that the magnitude of the curvature at a certain moment, is determined by the changes undergone by the difference in sensibility of the anterior and the posterior side from the beginning of the illumination up to a point of time which is separated from the chosen moment by the length of the latent period, i.e. the interval between the beginning of the illumination and the initiation of response; and 2nd that the direction of the curvature is determined by the sign of this difference in sensibility. It follows from this that an antiphototropic curvature will appear if the sensibility at the anterior side exceeds that at the

[1]) W. H. Arisz, Untersuchungen über den Phototropismus. Rec. d. trav. bot Néerlandais. Vol. XII. 1915.

posterior side. We will now ascertain under what conditions this will be the case.

As has been pointed out the direct result of the illumination is a decrease of sensibility, which is the greater according as the intensity of the illumination is stronger. It follows that under the influence of this factor the sensibility of the posterior side can never fall below that of the anterior side.

It has also been pointed out, however, that after some time the sensibility increases again at first slowly, then quicker and quicker, until after some time a velocity of increase is reached in agreement with the magnitude of the then existing sensibility. We should have to know now when the increase of the sensibility begins, and how long it lasts before it has reached its proper rate. This, however, has been investigated only in a single case. Meanwhile we are probably justified in supposing that these initial stages will be gone through the quicker according as the decrease of the sensibility has been greater. If this supposition is correct, then the increase of the sensibility in the anterior side must commence before that in the posterior side, at least when the sensibility in the latter is not wholly destroyed. If, therefore, the illumination is discontinued at a moment when the rate at which the sensibility increases has not yet reached its highest value in the posterior side, then the sensibility in the anterior side, if not too much below that in the posterior side, must soon prevail. This condition will then persist until the sensibility in both sides has reached once more its original value. If this view is right, the antiphototropic curvature can never be followed by a second normal curvature. In my own experiments, I may state such a second normal curvature could never be detected.

With highly intense light the antiphototropic curvature is not preceded by a normal one. If in this case the sensibility at the close of the illumination is still greater in the posterior side than in the anterior side, this difference does not persist long enough to express itself in a curvature of appreciable magnitude. Such antiphototropic curvatures are seen in the figures 11 and 12 of my paper cited above, which show the effects of an illumination respectively with 50 M.C. and with 250 M.C. They are most conspicuous with an exposition time respectively of $3^1/_2$ minutes and 40 seconds.

With a less intense light, on the other hand, the antiphototropic curvature is preceded by a normal one. The sensibility of the posterior side remains in this case long enough above that in the anterior side to express itself in a curvature.

Of course this curvature will be stronger according as the light

is weaker and the illumination lasts longer. Because the normal curvature has already shifted downwards a little way before the antiphototropic one becomes visible, the plants here assume the well-known *S*-shape. This case is illustrated in fig. 10 at an illumination of five minutes with 12.5 M.C.

With a light of still less intensity no more antiphototropic curvatures appear. We do find here after some time with the light-quantities that otherwise would have evoked an antiphototropic curvature, a considerable retardation in the velocity with which the curvature increases. When after the illumination these plants are not placed on the clinostat, they show the geotropic counter-curvature sooner than plants do that have been lighted for a longer or for a shorter period, and with which consequently this retardation does not occur. In fig. 9, which shows the effects of an illumination with $2^1/_2$ M.C. this curvature is seen at an exposition time of 10 minutes.

This proves that, in order to effect an antitropic curvature it is in the first place necessary to destroy the sensibility in the anterior as well as in the posterior side entirely or nearly so, a process dependent on the supplied energy; but in the second place to enable the sensibility in the anterior side to gain an advantage over the sensibility in the posterior side, a process that can take place only if the sensibility in the anterior side has vanished sooner than in the posterior side. If, therefore, the antiphototropic curvatures appear only in the manner described, they cannot appear with illuminations of very short duration. We shall presently inquire whether this conception is correct (see below).

That time is indeed an important factor in the process here discussed, is proved indubitably when we compare the effect of a unilateral illumination which is preceded by an omnilateral one with the effect of the same unilateral illumination, when it is followed by the omnilateral one. If it were only a question of the light-quantities in the antagonistic halves, the result would be the same in either case. This, however, is not so: whereas in the first case antitropic curvatures are seldom found and are never conspicuous, they appear regularly in the second case and are then generally well developed.

The second case is easy of explanation with my theory: As the sensibility at the anterior side is completely, or almost completely destroyed by the unilateral illumination, there will be a gradual increase of it during the omnilateral after-illumination; at the posterior side, on the contrary, the sensibility will still be more or less considerable at the end of the unilateral illumination, and it will,

therefore, still decrease during the omnilateral illumination. It follows, then, that it will reach in this side its lowest value only after some time has elapsed, and meanwhile it will have reached at the anterior side a higher value, and, as the advantage will remain with this side till the original sensibility in both sides will have been restored, it is to be expected that an antitropic curvature of some importance will arise. If a unilateral illumination is preceded by an omnilateral one (apart from those cases in which the omnilateral fore-illumination is very weak) the occurrence of an antiphototropic curvature would, on the contrary not be in keeping with my theory. I was, therefore, originally inclined to call in question the reports concerning the occurrence of antitropic curvatures under these circumstances, the more so as these reports are extremely doubtful: under almost the same conditions under which in one case curvatures were found, they could not be discovered in the other. In view of the results, to be recorded lower down, the possibility of their accuracy may, however, be admitted, at least in a certain number of cases.

If, besides the cause discussed here, there should be still another to call forth antiphototropic curvatures, it must manifest itself, as stated above, if the time-factor is eliminated as much as possible, i.e. if the duration of the illumination is shortened as much as possible. The advantage in the anterior side will then gradually decrease, and consequently the antiphototropic curvature also. To settle this question I made some experiments, which need not be described in detail, as the way in which they were performed did not differ from that used in my earlier experiments. Suffice it to say here that in some experiments the quantity of light amounted to 9000 M.C.S. and the period of illumination to $^3/_4$, 3, 12, 48, and 192 seconds, while in other experiments an amount of 12000 M.C.S. was used, and a period of illumination of 1, 4, 16, 64, and 256 seconds. After the illumination was concluded the plants were put on the clinostat on which they remained for 3 or 4 hours. Subsequently the curvatures were compared. It now appeared that everywhere antiphototropic curvatures came forth and that their magnitude differed only very little. It would seem indeed, as if the curvatures occurring with longer periods of illumination were generally slightly stronger, but the differences were so insignificant that they did not afford any certainty in this respect.

The experiments described go to show that there must be indeed still another factor by which antiphototropic curvatures can be brought about. Since the factor of time had been eliminated as much as possible, we must have to do here with a direct result of the

illumination. We must conclude then that the latter can occasion a retardation as well as an acceleration of the rate of growth. Most often we do not notice anything of the acceleration as it is concealed by the much more considerable retardation which is always present at the same time. Only when the value of the latter is the same on either side, a difference in acceleration will become visible in a curvature.

If the above is correct, then it must also be possible to obtain an antiphototropic curvature with plants that have first been submitted to omnilateral fore-illumination. If this illumination is such that a further retardation of the growth cannot be obtained by a prolongation of the illumination, then an antiphototropic curvature must be obtained with a comparatively weak after-illumination. However, it should seem that thus far the exact strength of the fore-illumination has not been hit upon in similar experiments. Anyhow, the antitropic curvatures obtained were slight, and required moreover a rather intense after-illumination.

In order to determine the retardation and the acceleration of the growth rate separately the processes should be severed. It is conceivable that we have to do here with the influence of light on two different photo-sensible systems. As the sensibility of the two systems would not be equal in the different parts of the spectrum in this case, an analysis could perhaps be performed by using light of different colours. It is very well possible, however that there is only one photo-sensible system, and that it is not the light itself but a product originating as a result of the illumination, which exerts its influence upon two different systems. In that case the separation of the two processes might be rendered possible by a study of the influence of the external conditions.

In connection with the last-mentioned supposition it is interesting to point out that according to the theory of Bose [1]) disturbances of equilibrium in the living organism manifest themselves in a contraction, which is accompanied by an expansion in the adjoining tissue. The water expelled with contraction, would heighten the turgor of the adjoining tissue and thus occasion there a temporary quickening of the growth. In our case, then, the antiphototropic curvature of the tip might result from the occurrence of a normal curvature more downwards. The investigations of Arisz make it probable that under the conditions of our experiments normal curvatures may have

1) J. C. Bose, Plant Response, Londen 1906; Life movements in plants. Vol. II. Calcutta 1919.

occurred in the basal portion. These curvatures are slightly noticeable in the photographs of my experiments. However, as they extend over a considerable portion of the coleoptile, they are not very conspicuous.

In Arisz's experiments the normal curvature of the basal portion was not accompanied by an antitropic curvature of the tip. This may be accounted for by the fact that the tip was protected here against the influence of the light. The turgescence of this part of the coleoptile, therefore, did not undergo any change, and was, therefore, perhaps unable to absorb the water expelled in the regions below.

According to this the occurrence of the antiphototropic curvature depends in this case on the sensibility being completely or almost completely destroyed in the tip, and only partially in the basal portion. With an equally intense illumination of the tip the appearance of an antitropic curvature could therefore not be expected when the basal portion of the coleoptile was not illuminated at all, or when it was illuminated so strongly that its sensibility was completely destroyed, as in neither case a normal curvature could appear. I have, however, not had an opportunity as yet to make these experiments.

Physiology. — *"On the Action of the Novocain on the Tonus of the Skeletal Muscle"*. By Dr. S. DE BOER. (Communicated by Prof. J. K. A. WERTHEIM SALOMONSON).

(Communicated at the meeting of October 29, 1921).

When a subcutaneous injection of 5 or 10 drops of 1 % novocain is administered to a frog, the muscles will soon become entirely atonic. The stimulus-threshold of the N. Ischiadicus is then still unchanged and likewise the sensibility of the skin. According to the hypothesis of ERICH MEYER and L. WEILER this atony is due to intoxication of the accessory nerve-endings of BOEKE. ALMS and also LILJESTRAND and MAGNUS ascribe it to intoxication of the sensibility of the muscle. The experimental investigation of FRANK and his co-workers corroborate MEYER and WEILER's conception.

When FRANK published his experience my inquiry had been in progress for some time. It is known that a small dosis of nicotin applied at the ingress of the nerve, produces a contracture. In order to obtain this effect by acting on the muscle-substance a much larger dosis must be used (LANGLEY). This result is not altered by a previous denervation: five weeks after cutting the N. Ischiadicus I succeeded in evoking contractures in denervated muscles on administration of a subcutaneous injection of a minimal dosis of nicotin.

When I injected into an intact frog first 5 or 10 drops of 1 % novocain subcutaneously, a complete atony of the skeletal muscles ensued after 15 minutes. Subsequently 10 drops of 1 % nicotin were given subcutaneously, which did not engender a trace of a contracture.

In another series of experiments I injected into the muscles of the hind-leg 3 drops of 1 % novocain to 1 cc of NaCl 0.65 %, by which these muscles lost their tonus. I then administered 10 drops of 1 % nicotin subcutaneously; the muscles of three limbs then displayed marked contractures; those of the hind-leg that had been poisoned with novocain remained quite flaccid. Similar experiments were carried out with a foreleg. The result was the same when I injected into the muscles of one of the non-poisoned limbs only 0.65 % NaCl solution.

When the excised rectus abdominis is placed in 40 gr. of NaCl 0.65 % and 1 drop of 1 % nicotin is added, a marked contracture will ensue. This will not take place, however, when first 40 gr. of NaCl 0.65 % + 4 gr. of 1 % novocain is allowed to act on it for 20 minutes.

When allowing this weak nicotin solution to act on the nervous aequator of a gastrocnemius without novocain, a marked contracture will appear, which will not be the case when the solution is made to act on the conical end. In order to achieve a contracture then a much stronger solution is required.

We conclude, therefore, that novocain can obviate contractures originating from the nervous aequator.

After this I used some substances that enhance the tonus, such as calcium-chloride and rhodan-sodium. For each of these I ascertained the weakened solution just yielding a distinct contracture of an excised skeleton-muscle. I then substituted the tenth part of such a solution by 1 % novocain. A frog's muscle was then submerged in it after the muscle had first been attached to a lever and after it had been for 20 minutes in 1 % novocain.

NaCNS yielded similar results to those obtained with nicotin. The contractures occurring after $CaCl_2$-poisoning, could not be obviated by a previous intoxication with novocain. Just as large doses of nicotin $CaCl_2$ would also yield contractures through an action on the tonus-substrate itself.

When in a frog hemisection of the med. oblongata proximal to the exit of the Nervus VIII is performed, there results a typical forced position. The ipsolateral foreleg is bent and adducted. The foreleg on the other side is abducted in a stretched position. The hind-legs display similar positions, but they are less pronounced. The head and the trunk are turned to the side of the lesion. Now when injecting into the muscles of the stretched and abducted foreleg two drops of 1 % novocain to 1 c.c. of NaCl 0.65 % the stretched position persists (in an uninjured frog such an injection produces a complete atony of the muscles of the foreleg). It appears then that the stretched position is evolved by a tetanic contraction, sustained under the influence of the ordinary cerebro-spinal innervation.

It appears, therefore, that an equally active dosis of novocain leaves the cerebro-spinal innervation intact, while the tonus of the skeletal-muscles is abolished by it. Now since novocain abolishes a muscle-contracture that has been evoked from the nervous aequator (receptive substance), we are justified in concluding that the tonus of the skeletal-muscles is destroyed by novocain through a poisoning of the receptive substance of the tonus-substrate. Moreover it has been proved once more that in the skeletal-muscles two kinds of contractions can be evoked, viz. clonic and tonic contractions.

Mathematics. — *"On the Path of a Ray of Light in the Field of Gravitation of a Single Material Centre."* By Prof. W. VAN DER WOUDE. (Communicated by Prof. J. C. KLUYVER).

(Communicated at the meeting of October 29, 1921).

1. Starting from the line element of the field of gravitation of a single material centre proposed at the same time by SCHWARZSCHILD [1]) and DROSTE [2]), I wish to demonstrate in this paper:

1. the path of any ray of light is (with allowable neglect) a hyperbola of which the sun is one of the foci; all these light paths have equal major axes;

2. by these geometrical data 8 hyperbolas are defined passing through 2 given points; from physical considerations however it appears easily that only one of the 8 hyperbolas connecting in this way the earth to an arbitrary star, is a path along which the light starting from the star, reaches the earth.

To this I add a remark on the determination of the magnitude of the deviation. [3])

2. The line element of SCHWARZSCHILD—DROSTE has the following expression:

$$ds^2 = \left(1 - \frac{\mu}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{\mu}{r}} - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2), \quad \ldots (1)$$

or, after the substitution

$$r = \varrho \left(1 + \frac{\mu}{4\varrho}\right)^2,$$

$$ds^2 = \left\{1 - \frac{\mu}{\varrho\left(1 + \frac{\mu}{4\varrho}\right)^2}\right\} dt^2 - \left(1 + \frac{\mu}{4\varrho}\right)^4 \{d\varrho^2 + \varrho^2(\sin^2\theta \, d\varphi^2 + d\theta^2)\} \quad (1')$$

[1]) K. SCHWARZSCHILD: Sitzungsberichte der Kön. Preuss. Akad. der Wissenschaften, 1916.

[2]) J. DROSTE: Het zwaarteveld van een of meer lichamen volgens de theorie van EINSTEIN. Thesis for the doctorate, Leiden, 1916.

[3]) For a prior discussion of the path of light in this field cf: W. DE SITTER: On EINSTEIN's Theory of Gravitation (Monthly Notices R. A. S., vol. LXXVI, p. 717).

A. S. EDDINGTON: Report on the Relativity Theory of Gravitation (p. 53—56).

H. VAN DER LINDEN: La Trajectoire d'un rayon lumineux dans le champ de gravitation d'EINSTEIN-SCHWARZSCHILD (Académie royale de Belgique, Bulletin, t. VI, 1920, p. 90—97).

Here μ is the mass of the centre of gravitation C.

Concerning the "space-coordinates" r (or ϱ), θ, φ, it is assumed that θ and φ have the same values as would be ascribed to them by an observer who made his observations and calculations in the conviction that space is Euclidian; for the rest neither r nor ϱ need be exactly equal to the distance R measured from C on Euclidian suppositions, but r and R or ϱ and R are supposed to be univalent functions of each other, so that $\frac{r}{R}$ and $\frac{\varrho}{R}$ differ so little from 1 that, at least for not too small values of R, $\left(\frac{r}{R}\right)^2$ and $\left(\frac{\varrho}{R}\right)^2$ may be neglected relatively to unity.

The units of length and time are assumed such that the light covers the unit of length in the unit of time, so that we may for instance consider 1 km. and $\frac{1}{3.10^5}$ sec. as those units; finally the mass μ is expressed in gravitation-units. If e.g. we consider the centre of the sun to be the centre of gravitation, we have $\mu = 1{,}47$, and already immediately outside the surface of the sun $\frac{\mu}{\varrho}$ will be very small; in the field outside the sun we may therefore, as a first approximation, neglect the second and higher powers of $\frac{\mu}{\varrho}$ and replace (1′) by

$$ds^2 = \left(1 - \frac{\mu}{\varrho}\right) dt^2 - \left(1 + \frac{\mu}{\varrho}\right) \{d\varrho^2 + \varrho^2 (\sin^2 \theta \, d\varphi^2 + d\theta^2)\} \quad . \quad (2)$$

3. In the space (2) the propagation of light occurs along a minimum line, i. e. along a line for which

$$ds = 0,$$

or for which

$$\left(1 - \frac{\mu}{\varrho}\right) dt^2 = \left(1 + \frac{\mu}{\varrho}\right) \{d\varrho^2 + \varrho^2 (\sin^2 \theta \, d\varphi^2 + d\theta^2)\};$$

as the nature of the field makes it at once clear that the path will lie in a "plane" through C, we may here put $\theta = \frac{1}{2}\pi$, so that for the path of the light we have:

$$\left(1 - \frac{\mu}{\varrho}\right) dt^2 = \left(1 + \frac{\mu}{\varrho}\right) (d\varrho^2 + \varrho^2 \, d\varphi^2) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is *assumed* that even now the path between 2 points A and B in three dimensional space may be found by making the condition that the first variation of the integral:

$$I = \int_A^B dt = \int_A^B \sqrt{\frac{1+\frac{\mu}{\varrho}}{1-\frac{\mu}{\varrho}}\,(d\varrho^2 + \varrho^2\, d\varphi^2)}$$

must be equal to zero. Neglecting the same quantities as before we may therefore say that we must determine φ as such a function of ϱ that

$$\delta \int_A^B \sqrt{\left(1+\frac{2\mu}{\varrho}\right)\left[1+\varrho^2\left(\frac{d\varphi}{d\varrho}\right)^2\right]}\, d\varrho = 0 \quad . \quad . \quad . \quad (4)$$

In order to solve this problem we point to another, which from a purely analytical point of view is equivalent to it. If in space, now thought to be Euclidian, a planet with the unit of mass moves according to NEWTON's law in the field of gravitation of the sun, according to the principle of least action its course is given by

$$\delta \int \sqrt{2\left(\frac{\mu}{\varrho}+h\right)\left[1+\varrho^2\left(\frac{d\varphi}{d\varrho}\right)^2\right]}\, d\varrho = 0,$$

where h is the constant of the living force.

From this it appears that the solution of (4)

$$\varphi = f(\varrho)$$

represents at the same time the orbit of a planet round the sun, where $h = \frac{1}{2}$, and that also the reverse is true.

Now each of these orbits of a planet is a conic section. However it would not be *quite* exact to say that the light path in three dimensional space is a conic section (unless we define a conic section in a non-Euclidian plane by a curve that has the same polar equation as a conic section; in differential geometry, however, the names ellipse and hyperbola are already given to different curves); by means of

$$\varphi = \varphi_1 \quad , \quad \varrho = \varrho_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

it is represented in a Euclidian plane by a conic section, where ϱ_1 and φ_1 are polar coordinates.

It is of some importance to remark that the formulas (5) represent the plane with the line element

$$ds^2 = \left(1+\frac{\mu}{\varrho}\right)(d\varrho^2 + \varrho^2\, d\varphi^2)$$

conformly on the Euclidian plane as the coefficients of the two fundamental forms are proportional.

If therefore it appears that in the image plane the tangent at B

to the light path AB makes with the radius vector CB a larger or a smaller angle than the straight line AB, it does not only follow from this that indeed according to EINSTEIN's theory out of B the point A will be seen in a different direction from that which was to be expected from the former theories, but also that the numerical value of this deviation may be read directly from the image.

Notwithstanding the objection just mentioned, confusion being excluded, we shall not hesitate henceforth to call the curve $\varphi = f(\varrho)$ in the EINSTEIN-plane a conic section; we only wanted to point out that for the final conclusion an appeal to the characteristic of conform representation is necessary.

4. Let us first consider once more the paths of material points moving with the unit of mass according to NEWTON's law in the field of gravitation of the sun C, thought to be Euclidian, while for all those paths the constant h has the same value; it is already certain that all those paths form a system of conic sections with a common focus C. It is further known that the semi major axis of such a conic section is determined by:

$$a = -\frac{\mu}{h}.$$ [1])

Hence all the conic sections have equal major axes; the sign of the axis indicates whether we have to do with an ellipse, a hyperbola or a parabola.

Applied to the problem in question this means:

The course of any ray of light is a hyperbola of which the sun is one of the foci; the length of the semi major axis is always equal to 2μ $(= \pm 3$ k.m.).

5. Now we shall determine the path of the light between 2 given points A and B.

With a view to this we describe out of C a circle γ with a radius 2μ and out of A and B two circles touching γ; the points of intersection of γ with AC and the produced part of AC are called resp. A'' and A'; B'' and B' are defined in the same way. Each point of intersection of one of the circles with centre A and one of the circles with centre B forms together with C the foci of a hyperbola through A and B, the major axis of which is 4μ. To begin with we find therefore 8 hyperbolas; let us now examine which of them gives a possible light path between A and B.

If S_4 is a point of intersection of the circles AA'' and BB', we have

[1]) Cf. e.g. P. APPELL: Traité de Mécanique rationelle, I p. 393.

$$AC - AS_4 = 4\mu = BS_4 - BC,$$

i.e. A and B lie on different branches of the hyperbola having C and S_4 for foci. This hyperbola is therefore a light path, but not one from A to B; neither is this the case with the two hyperbolas that have a focus in one of the points of intersection of the circles AA' and BB''.

The two circles AA'' and BB'' can have imaginary points of intersection; but it is also possible that these points of intersection are real. If on the latter supposition S_3 is such a point, we have

$$AC - AS_3 = 4\mu = BC - BS_3,$$

i.e. A and B lie on that branch of the hyperbola having C and S_3 for foci, that is not curved towards the sun, hence just on that branch that is not a light path.

If however S_1 and S_2 are the points of intersection of the circles AA' and BB', C and S_1 as well as C and S_2 are the foci of a hyperbola of which the same branch, the branch curved towards C, passes through A and B.

Of the 8 *hyperbolas through A and B which have C for a focus and of which the semi major axis is equal to* 2μ, *there are only two that give a possible light path; indeed, both are light paths, if only the branch on which A and B lie, does not intersect the sun.*

Here the question arises:

Let B be a point of the path of the earth and C the centre of the sun; can we place A in such a way that B is reached by 2 rays of light from A?

We call S_2 that point of intersection of the circles AA' and BB' that lies on the same side of AB as C; we put $\angle A'SB = \sigma$. It is clear that now CS_2 is less than CP, if P is the point of intersection of the tangents at A' and B' to γ; hence

$$CS_2 < 4\mu \operatorname{cotg} \tfrac{1}{2}\sigma.$$

The point S_2 lies therefore inside the sun (as $2\mu = \pm 3$) and the hyperbola with C and S_2 for foci will not be a light path connecting A and B, unless perhaps if σ is very small.

We shall therefore assume, in order to give CS_2 as great a value as possible, that A lies on the production of BC at infinite distance. We find

$$CS_2 < B'S_2 = \sqrt{8\mu \times 2BB'} = \pm 6 \times 10^4 \text{ km},$$

so that even now S_2 lies inside the sun. Accordingly: *It is impossible to see a point in* 2 *different directions out of a point of the path of the earth.*

Cryo-Biology. — *"Weitere physiologische Versuche mit niederen Temperaturen"* [1]). By P. GILBERT RAHM. (Versuche im physikalischen Laboratorium der Universität Leiden und der cryobiologischen Versuchsstation des Niederländischen Kälte-Vereins, Leiden Comm. Suppl. N°. 46*a*). (Communicated by Prof. J. P. KUENEN).

(Communicated at the meeting of October 29, 1921).

I. *Versuch mit flüssigem Wasserstoff.*

Zunächst wurden Versuche mit flüssigem Wasserstoff (— 253° C.) ausgeführt. Die Versuchstiere — es handelte sich wieder um Tiere der bryophilen Moosfauna d. h. Tardigraden, Nematoden und Rotatorien — waren mit dem Moos, in dem sie etwa drei Monate im Trockenschlaf zugebracht hatten, angefeuchtet und bald nach dem Erwachen zum aktiven Leben in einen Dewarschen Becher gelegt worden, der mit flüssigem Wasserstoff nach und nach gefüllt wurde. Die Tiere befanden sich in einem Reagenzgläschen, das mit einem leichten Wattestopfen abgedichtet ward. Der Versuch sollte 35 bis 40 Stunden dauern. Leider explodierte nach etwa 30 Stunden das Gefäss, in dem der flüssige Wasserstoff aufbewahrt worden war. Die sofortige Untersuchung der Moosprobe ergab ein negatives Ergebnis. Keines der Versuchstiere erwachte nach dem Auftauen zum Leben. Wahrscheinlich verschuldete die Explosion den Tod der Tiere und nicht die Kälteeinwirkung; denn bei früheren Versuchen blieb ein grosser Prozentsatz der Versuchstiere am Leben [2]), sofern die Kälte langsam einwirkte.

II. *Versuch mit flüssigem Helium.*

Versuchstiere: Moosfauna (Tardigraden, Nematoden, Rotatorien und Protozoen) im Trockenschlaf. Das Moos war von einem Dache in Monreal (Eifel) gesammelt und drei Monate lang lufttrocken aufbewahrt worden.

Verlauf des Versuches: Die Moosprobe wurde acht Tage lang im Vakuum (3 mm., zuletzt 24 Stunden 0,1 mm.) gelassen. Hierauf

[1]) P. G. RAHM, „Einwirkung sehr niederer Temperaturen auf die Moosfauna". These Proceedings, Vol. XXIII, pp. 235—248.

[2]) S. l. c.

füllte man Heliumgas ein, in dem die Moosprobe etwa 3 Tage lang verblieb. Am dritten Tage wurde mit der Abkühlung des Heliumgases begonnen. Nach dreistündiger Abkühlung ward das Heliumgas bei 4° K. flüssig. (= — 269° C.). Nach 2 Stunden und 10 Minuten wurde reduziert; die Temperatur betrug zeitweilig $1^1/_2$ bis 2° K. (= — 271 bis — 271,5° C.).

Das Heliumgas blieb 8 bis 9 Stunden flüssig. Hierauf stieg die Temperatur ganz langsam. Nach 12 Stunden war noch flüssiger Wasserstoff im Beirohr vorhanden. Die Moosprobe wurde sogleich herausgenommen und nahm in kürzester Zeit Zimmertemperatur an.

Ergebnis der Untersuchung: Leider konnte die Untersuchung Verhältnissehalber nicht sogleich an Ort und Stelle vorgenommen werden, sondern erst 14 Tage nach dem Versuch zur Ausführung gelangen.

Alle Versuchstiere — von Tardigraden auch die gepanzerten Echiniscus-Arten, Nematoden (Plectus-Arten), Rotatorien (Callidina-Arten) und Protozoen, deren Gattung leider noch nicht festgestellt werden konnte, erwachten bald nach dem Anfeuchten. Ein mehrmaliges Wiedereintrocknen und Anfeuchten ertrugen die meisten Tiere (bis 5 ×) schadlos [1]).

III. *Versuche mit flüssiger Luft.*

A. Mit *Coleopteren.* (Carabiden, Tenebrioniden und Curculioniden).

Lebende Coleopteren wurden den Dämpfen der flüssigen Luft ausgesetzt. Die Temperatur betrug anfangs — 20 bis — 30° C. Alle Versuchstiere gingen nach 5 bis 10 Minuten zu Grunde, obwohl alle Vorsichtsmassregeln beobachtet wurden. Es gelang auch nicht, die Tiere künstlich in Winterschlaf zu versetzen, wohl aus dem Grunde, weil die Versuchstiere zu dieser Zeit — die Experimente wurden im Sommer ausgeführt — noch keine Reservestoffzellen ausgebildet hatten und deshalb nicht die nötige Widerstandskraft besassen. Ein ausführlicher Bericht hierüber soll an anderer Stelle veröffentlicht werden.

B. *Mit Insekteneiern.*

Durch die Güte des Herrn Polak, Leiter des Insektariums der 'Artis' in Amsterdam, erhielt ich Eier von Dixippus morosus, der bekannten Stabheuschrecke, von Sphinx ligustri, dem Ligusterschwärmer, von Attacus edwardsii aus Assam und Samia cecropia aus Nordamerika.

[1]) Diese Versuche werden fortgesetzt und in der Zeitschrift für Allgemeine Physiologie, Fischer, Jena veröffentlicht.

Verlauf des Versuches: Die Eier wurden in einer Federspule, die mit einem Wattepfropfen leicht verschlossen war, langsam über den Dämpfen der flüssigen Luft abgekühlt. Die Temperatur betrug anfangs —10 bis —20° C. Nach je 5 Minuten wurden die Versuchseier tieferen Temperaturen ausgesetzt, indem die Federspulen langsam der Kältelösung genähert wurden. Erst nach $1\frac{1}{2}$ Stunden tauchten die Eier in die flüssige Luft ein. Die Temperatur betrug —190° C., die $3\frac{1}{2}$ Stunden einwirkte. Nach dieser Zeit wurden die Eier aus der Flüssigkeit gehoben, blieben aber noch den kalten Dämpfen ausgesetzt, bis sie nach und nach Zimmertemperatur annahmen.

Ergebnis der Versuche: Die Eier von Dixippus morosus haben zu 30% ihre Lebensfähigkeit bewahrt, doch bis heute (3 Monate nach dem Versuch) ist noch kein Tier geschlüpft. Aus den Lepidopteren-Eiern schlüpften bald nach dem Versuch Samia cecropia (etwa $\frac{1}{3}$). Die Eier waren 12 Tage alt, also dem Schlüpfen nahe. Die andern Lepidopteren Eiern scheinen alle abgestorben.

Zum Schlusse muss ich auch wieder an dieser Stelle dem Leiter des Kryogenen Instituts, Herrn Prof. Dr. Kamerlingh Onnes, für das überaus freundliche Entgegenkommen meinen allerherzlichsten Dank aussprechen. Nicht geringen Dank schulde ich ferner den Herrn Prof. Dr. Ehrenfest, Dr. Crommelin, van Eecke Herrn Mechaniker Flim und Herrn Spanjer.

Mathematics. — "*Explanation of some Interference-Curves of Uni-axial and Bi-axial Crystals by Superposition of Elliptic Pencils.*" (Second paper). By J. W. N. Le Heux. (Communicated by Prof. Hendrik de Vries).

(Communicated at the meeting of October 29, 1921).

In a former paper [1]), it is shown, that the interference-curves of some crystals — without regard to the isogyres — may be considered as the "moiré"-images of two concentric elliptic pencils, each containing the curves of intersection of the two parts of the successive wave-surfaces with the upper side of the crystal plate.

These wave-surfaces were supposed to be homothetic and so, the velocities in directions, normal to the wave-fronts, were uniform.

From experimental results it is seen, that this supposition explains the phenominae, observed in a polarisation-microscope, only at some distance from the centre, not immediately around it (fig. 1 and 2). For the caracteristic black cross, which appears around the centre with uni-axial and bi-axial crystals, is in those figures indistinctly to be seen.

It is proposed in this paper to give an account of some further experiments, which enable us to obtain the black cross in the centre with a fair accurancy and so to find some further conditions as to the elliptic pencils.

Consider first the figure of the hyperbolas (fig. 1). If the inner curves were not little circles, but ellipses, ending in short, coïncident, straight lines, this image would become much better. For in this way, a nearly sufficient effect was obtained by superposing two excentric sets of ellipses, as is shown in my first paper.

By causing two of the four pendulums of the former described apparatus to begin a very short time after the two others, the ending-curves of both unissons are short, coïncident, straight lines and the results are indeed more satisfying, but not yet wholly right.

There is another condition to be fulfilled, as was remarked with the following experiment.

The name of unisson was given — according to Lissajous — to the figure, described by two equal pendulums, with this extension, that the whole family of ellipses, described by altering the phase-difference from $\frac{\pi}{2}$ to 0, was also called unisson.

[1]) Proc. Royal Acad. XXIX, p. 1114—1117.

The beginning-curve was the circle, inscribed in the square upon the amplitudes, the ending-curve was a diagonal of this square.

A further extension is given by continuing the figure (the diagonal being run through), till another circle is described, with smaller radius, in consequence of the decrease of amplitudes, due to friction.

We will call this figure „continued unisson" — it shows a superposition of a larger and a smaller unisson and therefore interference-curves, which prove to be ellipses with their major axes in coïncidence with those of both unissons (fig. III).

This set of interference-ellipses is altered in a remarkable way, when the resistance of *one* of the composing movements increases. This may be obtained, with the apparatus used, by means of the screw-weight at the end of a ruler.

Successively, the ellipses are transformed into a flame-shaped image (fig. IV), then a black cross is formed around the centre (fig. V) and at last a set of hyperbolas appears (fig. VI), which is most plain, when the angle between the major axes of the beginning- and the ending-ellipse is between $\frac{\pi}{4}$ and $\frac{\pi}{2}$.

Increase of resistance of one of the composing movements induces a faster decrease of one of the amplitudes and so, the ellipses are no longer inscribed in squares but in rectangles. The major axis of each ellipse coïncides with the diagonal of the circumscribed rectangle. In this way, a rotation is caused around the centre, joined to the alteration in shape, already mentioned.

When the continued unisson is considered as a superposition of two common unissons, the rotations of the latter are in an opposite sense.

The principle of rotation arises immediately from the mathematical interpretation of the phenomenon.

In my first paper is already said, that in a superposition of two concentric pencils, the ellipses result from four vibrations.

Each pencil being given by

$$x = r\cos(\varphi + \alpha) + r\cos(\varphi + \gamma)$$
$$y = r\sin(\varphi + \beta) + r\sin(\varphi + \vartheta)$$

or by

$$x = 2r\cos\frac{\alpha-\gamma}{2}\cos\left(\varphi + \frac{\alpha+\gamma}{2}\right)$$
$$y = 2r\cos\frac{\beta-\vartheta}{2}\sin\left(\varphi + \frac{\beta+\vartheta}{2}\right)$$

and considering, that generally $\alpha-\gamma \neq \beta-\vartheta$, the ellipse with the

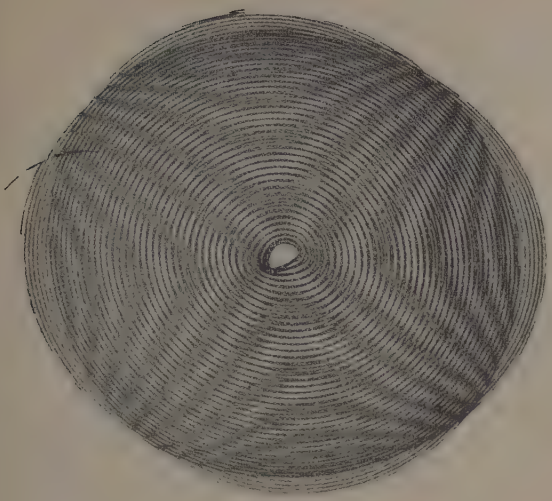

Fig. 1.

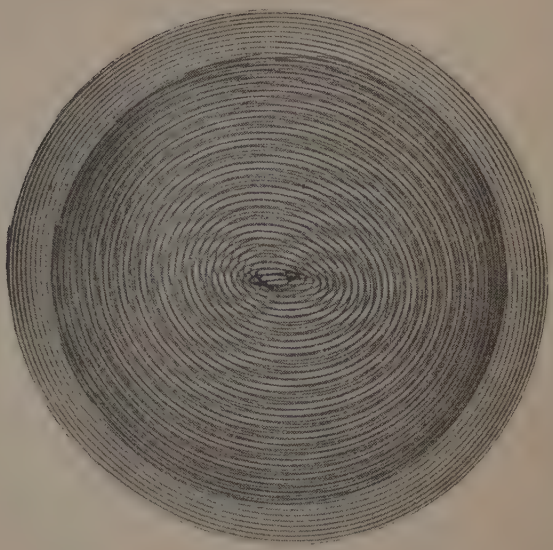

Fig. 2.

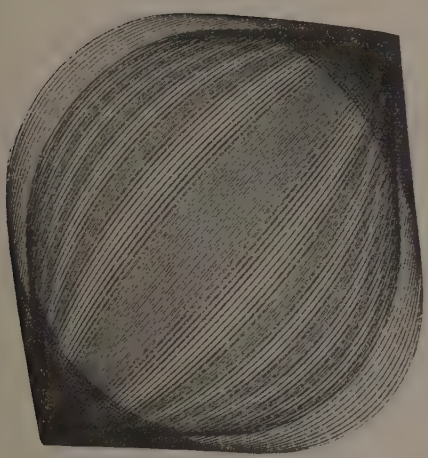

Fig. 3.

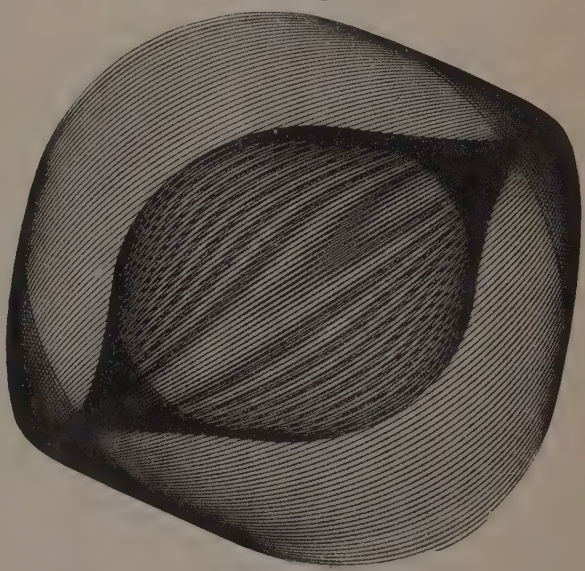

Fig. 4.

Fig. 5.

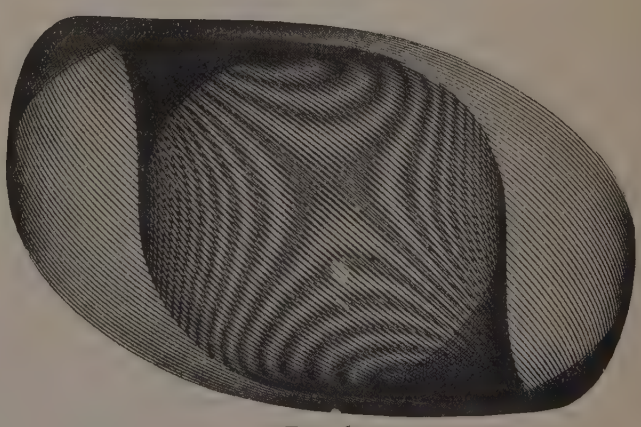

Fig. 6.

variable difference of phase $\frac{\alpha-\beta+\gamma-\vartheta}{2}$ is inscribed in a variable rectangle with sides $4r\cos\frac{\alpha-\gamma}{2}$ and $4r\cos\frac{\beta-\vartheta}{2}$ and therefore will rotate around the centre.

This rotation of the singular ellipses may not be confounded with the rotation of the whole unisson around the centre.

In this case, interference curves are also seen by superposing a unisson and the same figure after rotation — the image shows however slightly curved, parallel lines, cutting orthogonally the bisectrix of the angle between the axes, when the latter is about 15°.

At last, it is evident, that a sufficient result will be obtained in combining all the conditions, above mentioned, that is: The image of the hyperbolas proceeds from the superposition of two concentric pencils of ellipses (each ellipse resulting from four vibrations). The beginning-curves of these pencils are an ellipse and a circle, having common tangents in the extremities of the minor axis of the ellipse; the ending-curves are short, coincident, straight lines. The curves between show both a regular alteration in shape and a rotation.

The latter has an opposite sense for both pencils, but good results were also obtained in the case of rotation of one of the pencils only.

When these conditions are not observed, great differences become immediately visible.

So, the figure of the hyperbolas degenerates into a Maltese cross, when there is but a small angle between the ending-ellipses. The four arms of the cross point to the extremities of the major axes. Such a cross is also formed, when these ellipses (straight lines) are coïncident, but differ to much in magnitude.

The rotation of the ellipses being too fast, the "asymptotes" of the hyperbolas are curved in the same sense; when the two pencils have but a slight difference besides, the figure shows a pencil of curved radii.

If both ending-ellipses are coincident, but too large, the figure of the hyperbolas is exactly formed, but the upper and the lower part are translated in opposite sense.

The same remarks are applicable to the black cross around the centre in the image of the lemniscates.

The here described method of moiré figures to explain interference curves needs not to suppose (as is done with the commonly used method of isophase-surfaces of Bertin), that both broken normals to the wave-fronts follow the same way in the crystal.

Chemistry. — "*The Condition of Motion of the Molecules in Space.*" By Prof. J. BÖESEKEN. (In collaboration with Messrs. CHR. VAN LOON, DERX, and HERMANS).

(Communicated at the meeting of October 29, 1921).

In a paper on the configuration of the tartaric acids (BÖESEKEN and COOPS) [1]) the conclusion was drawn from the behaviour of these acids and their esters towards boric acid, that the carboxyl groups, both when substituted and when not substituted, were as far apart as possible.

The amides and ethyl-amides, however, presented deviations, so that this simple assumption did not suffice. The great influence which these substances have on the conductivity of boric acid, points to a favourable situation of the hydroxyl groups with regard to each other, which can only satisfactorily be accounted for, when there are *attractive* actions between these groups and the substituted or non-substituted amide groups.

Further for a comparatively large number of α-glycols no influence on the conductivity of the boric acid was found; these OH-groups must, therefore, lie far apart, which could again be explained by mutual repulsion. At any rate it appeared from these investigations that atoms that are not directly bound to each other, also exert an action on each other, and we may expect that the state of equilibrium of a molecule will be the result of all the forces in that molecule, both the attractive and the repulsive ones.

I have already pointed out [2]) that every change in the position of the atoms, however slight, must give rise to re-arrangements, through which the position of *all* the parts of the molecule will not remain exactly the same.

In the molecule-halves of active and inactive tartaric acid e.g. the comparable groups will no longer be perfectly identical; hence the principle of the optical superposition cannot be valid.

In consequence of the action of these forces, molecules which up to now have been considered as symmetrical, as the anti-tartaric

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[1]) Versl. Kon. Ak. Wet. **29**, 368 (1920). Has not yet appeared in English.
[2]) „ „ „ „ **29**, 562 (1920). „ „ „ „ „ „

acid, will no longer have mirror images that cover each other, which can easily be demonstrated by means of the well-known carbon models.

Only two positions of the anti tartaric acid are symmetrical, viz. that in which the equal groups lie on the same side of the axis of the central C-atoms as close as possible to each other and that in which they lie on both sides as far from each other as possible. It would be quite accidental, that one of these positions should be the most stable state of equilibrium under all circumstances.

This seems in conflict with experience. We should, however, bear in mind the following considerations. In view of the limited number of isomers it has already long been assumed that the molecule halves can move round the single bonds as axes in opposite direction or with different velocities.

When we supplement this necessary condition with the hypothesis that these movements also continually take place in this sense that the most stable position of all the positions possible at the moment will occur most frequently, and that, therefore, the molecule executes irregular rotations or oscillations round this position, there is no longer any contradiction with experience.

When we consider these rotations round the bond of the central carbon atoms of the anti-tartaric acid, we see that in one full rotation of one part with regard to the other, the molecule will twice pass a symmetric position.

At these moments the molecule is optically inactive, and since now the chance has become equally great that the stable asymmetric-position will be reached by a movement in the direction of the hands of a clock or in opposite direction, a continual racemisation will take place.

Though the state of equilibrium is asymmetric, we shall probably never observe the optical activity in liquid or dissolved state.

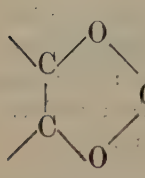
This dynamic representation, which is forced upon us by the inactivity of the anti-tartaric acid, applies, of course, to all saturated molecules. Our observations of the behaviour of the α glycols towards boric acid and acetone can be explained most simply in this way.

The saturated non-cyclic α-glycols do not increase the conductivity of the boric acid; we have drawn the conclusion from this that the hydroxyl-groups are not favourably situated, and have accounted for this by the assumption that the OH-groups repel each other, through which they can get as far as possible from each other owing to the mobility of the molecule.

According to the dynamic representation this position will be the

most stable state of the molecule, but all the other positions will also be realized, albeit less frequently, and the more rarely as the state deviates more considerably from the most stable.

Of the α-glycols there are cyclic acetone-compounds known, which are formed with excess of acetone, to which a little HCl or H_2SO_4 has been added, and which are stable, when there is not a trace of these last catalysts present.



There is every reason to assume that there is the greatest chance that these compounds are formed when the OH-groups are situated in the plane with the C-atoms to which they are bound, and on the same side of them. When the OH-groups repel each other, this will seldom take place, but by taking a very great excess of acetone, we shall yet be able to obtain a pretty large quantity of the cyclic compound, which can be retained by removal of the catalyst, and which can be freed from the acetone by fractional distillation.

When the OH-groups cannot get into the most favourable position, or near it, as in some cyclic trans 1.2-diols, where the free rotation through the ring is prevented, no acetone-compounds will be formed. This now has been found, and this renders a firm support to this dynamic representation (see below).

The saturated glycols can, however, always assume the most favourable position, though it be only rarely; the relative frequency of this case may be judged from the state of the equilibrium:

$$\text{glycol} + \text{acetone} \gtrless \text{acetone compound} + \text{water}.$$

By determination of the constant of equilibrium:

$$K_t = \frac{C_{\text{glycol}} \times C_{\text{acetone}}}{C_{\text{comp.}} \times C_{H_2O}}$$

a measure is supplied for this frequency, and so also a measure for the situation of the hydroxyl groups in the most stable position of the molecule.

Here follows a summary of the constants of equilibrium of some glycols:[1])

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[1]) In his thesis for the doctorate (Delft 1919, p. 59) "over de Stereochemie der cyclopentaandiolen 1.2 en hydrindeendiolen 1.2". Mr. CHR. VAN LOON was the first to point out that the study of these equilibria would probably bring to light minuter differences in the configuration of the molecules than the boric acid method. The measurements were carried out by Mr. P. HERMANS.

	$K_{18°}$
ethylene glycol	0.14
propane diol 1.2	0.44
„ 1.3	0.026
α-monochlorhydrine	0.28
glycerine	0.77
cis-cyclohexanediol 1.2	0.15
trans- „ 1.2	0.000

The study of the cyclic glycols has supplied an important support to the efficiency of the dynamic representation.

In the cyclo-pentane-diols the five carbon atoms lie in one plane almost without tension when we assume that the directions of affinity make an angle of 109°28′ with each other. In the cis-isomers the OH-groups lie almost in one plane with the C-atoms, to which they are bound, hence very favourably for boric acid and acetone, as has appeared from VAN LOON's investigation.

In the trans-diols the position is very unfavourable. In consequence of the five-ring the position of these OH-groups can never become favourable without the ring being broken. Indeed there was no question of the formation of an acetone compound.

The absence of the acetone-compound in this trans-diol and in the trans-hydrindene diol 1.2 is also the proof that when the OH-groups are prevented from getting from time to time into or almost into one plane, no five-ring is formed. (see above).

At the time it was found that cis-cyclohexane diol 1.2 has no influence on the conductivity of the boric acid. This had led me to conclude already then that this six-ring must be assumed to have a certain suppleness, through which the hydroxyl groups would get somewhat further from each other than would have been expected, if the six atoms of the ring lay permanently in a plane.

The equilibrium: glycol + acetone $\gtrless$ acetone compound + water corroborated on one side this view, as its position is almost as unfavourable as with the ethylene glycol (see above).

On the other hand it corroborated our dynamic view: The acetone compound is formed; hence the hydroxyl groups must now and then lie in or almost in the same plane as the adjacent C-atoms.

A complete rotation, as is the case with the saturated non-cyclic glycols, is not possible here, as this would mean a continual transition from cis- into trans-diol, and vice-versa, which does not take place; besides, the trans-diol gives no trace of an acetone compound.

How can we now form an idea of the movement in the cyclohexane diol?

In 1890 an extensive study was published by SACHSE [1]) on the position of the C-atoms in the saturated ring systems starting from the premise that the directions of affinity form angles of 109°28′ with each other. He proves for the three-, four-, and five-rings that the atoms must be situated in one plane, and that a tension is inevitable, which is, however, insignificant for the five-ring. (VON BAYER's ring tension). He shows for the six-ring that the molecule can escape a tension in two ways. First of all the C-atoms can lie three in one plane, and three in another, so that the bonds between the C-atoms form a zig-zag line. It is, however, also possible that four atoms lie in one plane, and two (e.g. 1 and 4, 2 and 5 or 3 and 6) outside it.

To bring the molecule out of the first position, a certain resistance must be surmounted; the second position is pliable.

It has been pointed out on different sides that these positions of the molecule could not form the image of the stable situation of the atoms, because the number of isomers, and particularly of optical isomers would then have to be much greater.

ASCHAN [2]), however, remarks, that these positions *are* possible when they are considered as the successive phases of a system in motion.

This view, indeed, has been confirmed by our boric acid measurements and the study of the acetone-compounds.

When we accept the view that the molecules of the six-rings move through space as undulatory surfaces, the OH-groups of the cis-diol get from time to time into the same plane with and on the same side of the C-atoms to which they are attached, and the frequency of this occurrence will depend on the most stable state of the molecule.

It is clear that in these undulations repeatedly a symmetric position is passed, so that, though the most stable situation is an asymmetric one, the separation of optical isomers in liquid state is not very well possible; for racemisation will continually set in for

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[1]) Berichte **23,** 1363 (1890). Zeitschr. f. physik. Chemie **10,** 228 (1890).

[2]) "Chemie der Alicyclischen Verbindungen", pag. 328—331.

the same reason as with the movements of the anti-tartaric acid. In the trans-cyclohexane diol the OH-groups always remain very far apart; indeed, we have not been able to obtain an acetone-compound, nor has an influence been observed on the conductivity of the boric acid.

Another confirmation was furnished by the cis- and transcycloheptane diols.

When SACHSE's calculations are applied to them, the OH-groups, both in the cis- and in the trans-diol, get in their movements into, or nearly into the same plane with and on the same side of the C-atoms, which they adjoin. Now acetone compounds have actually been obtained, and increase of the conductivity of boric acid has been observed, *both in the cis- and in the trans-diol*; accordingly our measurements are in perfect harmony with what is to be expected in the movements of this system as undulatory surface.

According to this view there is, therefore, no tension in the six- and sevenring-systems, even less than in the five-ring, and in fact, we knew this already from the measurements of the heats of combustion of the cyclo-paraffins. The very accurate determinations by STOHMANN, ROTH, and ZUBOW leave no room for doubt that the increase of the heat of combustion per CH_2-group of C_5H_{10}, C_6H_{12}, C_7H_{14} does not amount to more than has been found in the paraffins, which ought to have been the case with increasing tension.

This result seems to be in conflict with the experiences obtained about the ring-closures. When there is no ring-tension present in the saturated rings with more than six atoms, why are not they formed as easily as the five-rings, and why are they also clearly less widely spread in nature?

This more difficult formation is, indeed, strikingly illustrated by the very low constant of equilibrium of the system propane diol 1.3 + acetone, especially when compared with the corresponding propane diol 1.2 + acetone, the difference of which rests on six-ring closure against five-ring closure (see above).

An answer is easy to give also here.

The probability that the two OH-groups in the propylene glycol 1.2 assume a favourable position with regard to the acetone molecule is so much greater than for the 1.3 isomer. In the propane diol 1.2 a rotation of the molecule parts round a single bond suffices, whereas in the propane diol 1.3 the paths of the OH-groups are much more involved and intricate, so that in the same space of time they will certainly assume a favourable position much less frequently.

We may summarise the result of this study as follows:

1. Also atoms that are not directly bound to each other, exercise an action on each other in the same molecule.

2. The saturated non-cyclic molecules execute, among others, movements in which the parts of the molecule revolve in opposite direction or with different velocities round the single bonds as axes.

In case of not uniform load, as is the case in by far the most molecules (excepted are only H_2, O_2, N_2 etc. C_2H_6, C_2Cl_6, perhaps oxalic acid, etc.) the movements are irregular, because the most stable position of the atoms will be passed most frequently.

3. In the saturated ring-shaped molecules with six and seven carbon atoms the ring-forming atoms are not fixed in one plane, but they lie tension-less in a curved surface, which travels through space in undulatory movements.

Delft, October 1921.